27. *On the Gaps between the Consecutive Zeros of the Riemann Zeta Function*

By Akio FUJII
Department of Mathematics, Rikkyo University

(Communicated by Shokichi IYANAGA, M. J. A., April 12, 1990)

§ 1. **Introduction.** Let γ_n be the n-th positive imaginary part of the zeros of the Riemann zeta function $\zeta(s)$. We have shown in [1] and [2] that for each integer $k\geq 1$ and for $T>T_0$,

$$C_1\frac{T\log T}{\log^k T}\leq \sum_{T\leq\gamma_n\leq 2T}(\gamma_{n+1}-\gamma_n)^k\leq C_2\frac{T\log T}{\log^k T},$$

where C_1 and C_2 are some positive constants. The implicit constant C_2 might be large. The purpose of the present article is to get an explicit C_2 (for the case $k=2$) under the assumption of the Riemann Hypothesis. We shall prove the following theorem.

Theorem 1. *For $T>T_0$, we have*

$$\sum_{\gamma_n\leq T}(\gamma_{n+1}-\gamma_n)^2\leq 9\cdot\frac{2\pi T}{\log\dfrac{T}{2\pi}}.$$

We shall prove this theorem as an application of the following mean value theorem which has been proved in [5]. We put

$$S(t)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+it\right)$$

and

$$F(a)=F(a,T)=\left(\frac{T}{2\pi}\log\frac{T}{2\pi}\right)^{-1}\sum_{0<\gamma,\gamma'\leq T}\left(\frac{T}{2\pi}\right)^{ia(\gamma-\gamma')}\frac{4}{4+(\gamma-\gamma')^2},$$

where γ and γ' run over the imaginary parts ($\neq 0$) of the zeros of $\zeta(s)$.

Theorem 2. *Suppose that $0<\Delta=o(1)$. Then we have for $T>T_0$,*

$$\int_0^T(S(t+\Delta)-S(t))^2dt$$
$$=\frac{T}{\pi^2}\left\{\int_0^{\Delta\log(T/2\pi)}\frac{1-\cos(a)}{a}da+\int_1^\infty\frac{F(a)}{a^2}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da\right\}+o(T).$$

In fact, we shall use it in the following form.

Corollary. *Suppose that $T>T_0$ and $1/\log(T/2\pi)\leq\Delta=o(1)$. Then we have with $|\theta|\leq 1$ and the Euler constant C_0,*

$$\int_0^T(S(t+\Delta)-S(t))^2dt=\frac{T}{\pi^2}\log\left(\Delta\log\frac{T}{2\pi}\right)$$
$$+\frac{T}{\pi^2}\left(C_0-\frac{\sin(\Delta\log(T/2\pi))}{\Delta\log\dfrac{T}{2\pi}}+2\theta\left(\frac{1}{\left(\Delta\log\dfrac{T}{2\pi}\right)^2}+2\right)+o(1)\right).$$