22. On the Distribution of the Zeros of the Riemann Zeta Function in Short Intervals

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(Communicated by Shokichi IYANAGA, M. J. A., March 12, 1990)

§1. **Introduction.** Let $\zeta(s)$ be the Riemann zeta function. In [2] and [5], the author has shown that for $T>T_0$, $0<\Delta\ll 1$ and for each integer $k\geq 1$,

$$\int_0^T (S(t+\Delta)-S(t))^{2k}dt$$
$$=\frac{(2k)!}{(2\pi)^{2k}k!}T(2\log(2+\Delta\log T))^k+O(T(\log(2+\Delta\log T))^{k-(1/2)}),$$

where we put $S(t)=(1/\pi)\arg\zeta(1/2+it)$ as usual. This formula has been proved to be powerful in the theory of the Riemann zeta function (cf. [7], [9] and [10], for example). Recently, some attentions have been paid to it from the view point of the comparison with the distribution of the eigenvalues of the Gaussian Unitary Ensembles (cf. [1], [3], [7] and [12]).

In the present article, we shall assume the Riemann Hypothesis and refine the above result for $k=1$ as follows. To state our result we put

$$F(\alpha)\equiv F(\alpha,T)\equiv\left(\frac{T}{2\pi}\log\frac{T}{2\pi}\right)^{-1}\sum_{0<\gamma,\gamma'\leq T}\left(\frac{T}{2\pi}\right)^{i\alpha(\gamma-\gamma')}\frac{4}{4+(\gamma-\gamma')^2},$$

where $\alpha\geq 0$, γ and γ' run over the imaginary parts ($\neq 0$) of the zeros of $\zeta(s)$.

Theorem. *Suppose that $0<\Delta=o(1)$. Then we have for $T>T_0$,*

$$\int_0^T (S(t+\Delta)-S(t))^2dt$$
$$=\frac{T}{\pi^2}\left\{\int_0^{\Delta\log(T/2\pi)}\frac{1-\cos(\alpha)}{\alpha}d\alpha+\int_1^\infty\frac{F(\alpha)}{\alpha^2}\left(1-\cos\left(\alpha\Delta\log\frac{T}{2\pi}\right)\right)d\alpha\right\}+o(T).$$

We shall prove this by applying Goldston [8] while we have applied Selberg [13] to prove our previous mean value theorem described above.

Some applications of this theorem will be discussed in the forthcoming paper [6].

§2. **Proof of Theorem.** We shall use first the following Goldston's explicit formula for $S(t)$ (cf. p. 157 of [8]). For $t>1, t\neq\gamma$, $x=(T/2\pi)^\beta$ and $0\leq\beta\leq 1$,

$$S(t)=-\frac{1}{\pi}\sum_{n\leq x}\frac{\Lambda(n)\sin(t\log n)}{\sqrt{n}\log n}f\left(\frac{\log n}{\log x}\right)$$
$$+\frac{1}{\pi}\sum_{\gamma}h((t-\gamma)\log x)+O\left(\frac{1}{t\log^2 x}\right)+O\left(\frac{\sqrt{x}}{t^2\log^2 x}\right)$$
$$=-A(t)+B(t)+O\left(\frac{1}{t\log^2 x}\right)+O\left(\frac{\sqrt{x}}{t^2\log^2 x}\right),\ \text{say},$$