78. On Certain Real Quadratic Fields with Class Numbers 3 and 5

By Shin-ichi KATAYAMA*) and Shigeru KATAYAMA**)

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Let d be a square-free integer of the form $d=r^2+4$ or r^2+1 $(r\in \boldsymbol{N})$. Recently H.K. Kin, M.-G. Leu and T. Ono proved in [4] that there exist 11 real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ with class number one with at most one exception of d. In [5] Leu proved further that there exist 16 real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ of class number 2 with at most one exception of d.

In this paper, we shall consider the fields of the same kind with class numbers 3 and 5. Our main theorem is obtained by using almost the same methods as in [4], [5] and by the help of a computer.

We denote the class number of the quadratic field $\boldsymbol{Q}(\sqrt{d})$ by $h(d)$, which we shall abbreviate to h, if there is no fear of confusion. By the genus theory of quadratic fields, we have the following lemma.

Lemma 1. *If d is square-free and $d=r^2+4$ or r^2+1 and $h(d)$ is odd, then d is a prime.*

In the following, we restrict ourselves to the case $h(d)=3$ or 5. Therefore d is a prime and is denoted by p.

Then p should be expressed in the form $p=m^2+4$ (m: odd) or $p=4m^2+1$ $(m\in \boldsymbol{N})$. We note here that $p\equiv 1(4)$ and the fundamental unit $\varepsilon_p=(t+u\sqrt{p})/2$ of $\boldsymbol{Q}(\sqrt{p})$ is given with $t=m$, $u=1$, in the case $p=m^2+4$, and given with $t=4m$, $u=2$ in the case $p=4m^2+1$.

Let χ_p be the Kronecker character belonging to $\boldsymbol{Q}(\sqrt{p})$ and $L(s,\chi_p)$ be the corresponding L-series. Then by Theorem 2 of [6], for any $x\geqq 11.2$ and $p\geqq e^x$, we have

$$L(1,\chi_p)>\frac{0.655}{x}p^{-1/x}$$

with one possible exception of p. From class number formula, we obtain

$$h(p)=\frac{\sqrt{p}}{2\log\varepsilon_p}L(1,\chi_p)>\frac{0.655}{x}\,\frac{\sqrt{p}\,p^{-1/x}}{2\log(u\sqrt{p})}$$
$$=\frac{0.655}{x}\,\frac{p^{(x-2)/2x}}{2\log u+\log p}>\frac{0.655\,e^{(x-2)/2}}{x(x+3)}.$$

Since $f(x)=\dfrac{e^{(x-2)/2}}{x(x+3)}$ is a monotone increasing function for $x\geqq 11.2$, we have

$$h(p)>0.655f(17)=3.48\cdots>3\ (x\geqq 17),$$
$$h(p)>0.655f(18)=5.16\cdots>5\ (x\geqq 18).$$

*) College of General Education, Tokushima University.
**) College of General Education, Tokushima Bunri University.