# 59. *Recurrent Fuchsian Groups whose Riemann Surfaces have Infinite Dimensional Spaces of Bounded Harmonic Functions*

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**§1. Introduction and statement of results.** A Fuchsian group $\Gamma$ acts on both the unit disc $\boldsymbol{D}$ and on the unit $\boldsymbol{S}^1$. Such a group is said to be recurrent if, for any positive measure subset $A \subset \boldsymbol{S}^1$, $\sharp\{\gamma \in \Gamma : m(A \cap \gamma A) > 0\} = \infty$. Such groups have been considered as a subject of study in their own right primarily since the appearance of Dennis Sullivan's profound paper [9].

The function theory corresponding to such groups is not yet understood. For example, if $\mathfrak{R} = \boldsymbol{D}/\Gamma$ (we will use this notation throughout this note), the structure of the spaces of bounded harmonic or bounded holomorphic functions are not yet clear. Taniguchi constructed examples of Fuchsian groups such that the space of bounded harmonic functions on $\mathfrak{R}$, $HB(\mathfrak{R})$, is finite dimensional [11].

Now bounded harmonic functions on $\mathfrak{R}$ arise from integrating $\Gamma$-invariant measurable functions on $\boldsymbol{S}^1$ against the Poisson kernel and projecting down to $\mathfrak{R}$ from $\boldsymbol{D}$. We define harmonic measure class on $\mathfrak{R}$ to be the $\sigma$-algebra of $\Gamma$-invariant measurable subsets of $\boldsymbol{S}^1$, with the measure $m_p$ on $\boldsymbol{S}^1$ associated to a point $p \in \boldsymbol{D}$ just the visual measure from $p$ with respect to the hyperbolic metric on $\boldsymbol{D}$.

It is easy to see that the notions of positive measure and zero measure sets are well-determined in this measure class (although the measure of a positive measure set is only defined if it is 1 or 0). Furthermore, the notion of an atom in this harmonic measure class is well-defined as an ergodic component with positive measure. Given the definition of $O_{HB}^{\infty}$ from [2, pp. 119–128], one easily deduces.

**Lemma 1.** *$\mathfrak{R} \in O_{HB}^{\infty}$ if and only if $\Gamma$ decomposes $\boldsymbol{S}^1$, up to measure zero, into a union of positive measure ergodic components for its action.*

The proof is left to the reader. Given Lemma 1 and his examples, Taniguchi proffered

**Conjecture** [1, p. 4]. *If $\Gamma$ is a recurrent Fuchsian group then the Riemann surface $\mathfrak{R} = \boldsymbol{D}/\Gamma$ is in $O_{HB}^{\infty}$.*

This note concern two points. The first is that the conjecture is false. Let $K$ denote the usual middle thirds Cantor set, and $\mathfrak{R}_K$ denote the Riemann

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