5. On the Total Variation of Argument $f(z)$ Whose Derivative Has a Positive Real Part

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1. Introduction. Let R denote the class of functions which are analytic and satisfy $Ref'(z)>0$ for $|z|<1$ and are normalized by $f(0)=0$ and $f'(0)=1$.

Noshiro [3] and Warschawski [4] showed that $Ref'(z)>0$ is a sufficient condition for the univalence of $f(z)$ in any convex domain.

MacGregor [2] investigated the class of functions which belong to R and obtained many interesting results.

It is the purpose of the present paper to obtain the total variation of argument $f(z)$ whose derivative has a positive real part.

2. Preliminaries. Lemma 1. *Let $f(z) \in R$, then*

$$\left|\frac{z}{f(z)}\right| \leqq \frac{2}{r}\log(1+r)-1$$

where $0<|z|=r<1$.

We owe this lemma to [2, Theorem 1].

Lemma 2. *Let $f(z) \in R$, then*

$$\int_0^{2\pi} |f'(z)|\,d\theta \leqq 2\pi + 4\log\frac{1+r}{1-r}$$

where $|z|=r<1$.

We owe this lemma to [1, p. 482].

3. Statement of result. Theorem. *Let $f(z) \in R$, then*

$$\int_0^{2\pi}\left|\mathrm{Re}\frac{zf'(z)}{f(z)}\right|d\theta \leqq 2\pi + 4\log\frac{1+r}{1-r} \tag{1}$$

where $|z|=r<1$.

Proof. From Lemmas 1 and 3, we easily have

$$\begin{aligned}\int_0^{2\pi}\left|\mathrm{Re}\frac{zf'(z)}{f(z)}\right|d\theta &\leqq \int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|d\theta \leqq \left(\frac{2}{r}\log(1+r)-1\right)\int_0^{2\pi}|f'(z)|\,d\theta \\ &\leqq (2\log(1+r)^{1/r}-1)\left(2\pi+4\log\frac{1+r}{1-r}\right) \\ &\leqq 2\pi + 4\log\frac{1+r}{1-r}\end{aligned}$$

where $0<|z|=r<1$.

For the case $r=0$, the estimation (1) is true.

This completes our proof and (1) implies that

$$\int_0^{2\pi}\left|\mathrm{Re}\frac{zf'(z)}{f(z)}\right|d\theta = 0\left(\log\frac{1}{1-r}\right) \qquad as\ r\to 1.$$