110. A Poincaré-Birkhoff-Witt Theorem for the Quantum Group of Type A_N

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Introduction. To each complex semisimple Lie algebra $\mathcal{G}$, Jimbo [4] and Drinfeld [1, 2] associated a Hopf algebra $U_q\mathcal{G}$ with a nonzero complex parameter q. This Hopf algebra, which is called a *quantum group* by Drinfeld [2], can be considered as a natural q-analogue of the universal enveloping algebra $U\mathcal{G}$ of $\mathcal{G}$. In this note, we give an explicit linear basis of $U_q\mathcal{G}$ when $\mathcal{G}=sl_{N+1}(C)$. This result can be considered as a natural q-analogue of the Poincaré-Birkhoff-Witt theorem for $U_q sl_{N+1}(C)$. As a corollary of this, for $q(q^8-1)\neq 0$, we can show that $U_q sl_{N+1}(C)$ is a left (right) Noetherian ring, and that $U_q sl_{N+1}(C)$ has no zero divisors $\neq 0$. We also give a triangular decomposition of a general quantum group $U_q\mathcal{G}$. This is used in proving our theorem. Details which are omitted here will be published elsewhere.

1. Let F be a field and $F^\times$ the set of nonzero elements of F. Let $(a_{ij})_{1\leq i,j\leq N}$ be the Cartan matrix of type A_N. For $q\in F^\times$ such that $q^4\neq 1$, let $U_q sl_{N+1}$ be the associative F-algebra with 1 with generators $e_i, f_i, k_i^{\pm 1}$, $1\leq i\leq N$, and relations

$$k_i k_i^{-1}=k_i^{-1}k_i=1, \qquad k_i k_j=k_j k_i \tag{1.1}$$

$$k_i e_j k_i^{-1}=q^{a_{ij}}e_j, \qquad k_i f_j k_i^{-1}=q^{-a_{ij}}f_j \tag{1.2}$$

$$e_i f_j - f_j e_i=\delta_{ij}\frac{k_i^2-k_i^{-2}}{q^2-q^{-2}} \tag{1.3}$$

$$\begin{aligned} &e_i^2 e_j-(q^2+q^{-2})e_i e_j e_i+e_j e_i^2=0 && \text{for } |i-j|=1,\\ &e_i e_j-e_j e_i=0 && \text{for } |i-j|\geq 2 \end{aligned} \tag{1.4}$$

$$\begin{aligned} &f_i^2 f_j-(q^2+q^{-2})f_i f_j f_i+f_j f_i^2=0 && \text{for } |i-j|=1,\\ &f_i f_j-f_j f_i=0 && \text{for } |i-j|\geq 2. \end{aligned} \tag{1.5}$$

For $1\leq i<j\leq N+1$, we define inductively the elements e_{ij}, f_{ij} of $U_q sl_{N+1}$ by

$$e_{i\,i+1}=e_i, \qquad f_{i\,i+1}=f_i,$$

$$e_{ij}=qe_{ij-1}e_{j-1j}-q^{-1}e_{j-1j}e_{ij-1} \qquad \text{for } j-i\geq 2,$$

$$f_{ij}=qf_{ij-1}f_{j-1j}-q^{-1}f_{j-1j}f_{ij-1} \qquad \text{for } j-i\geq 2.$$

(The elements e_{ij}, f_{ij} were first introduced by Izumi [3], and Jimbo independently.)

Let $\Lambda_N=\{(i,j)\in Z\times Z \mid 1\leq i, j\leq N+1\}$. Define the lexicographic order $<$ on Λ_N by

$$(i,j)<(m,n) \quad \text{if and only if} \quad i<m \quad \text{or} \quad i=m,\ j<n.$$

Now we can state our theorem.

Theorem. *Let $q\in F^\times$ such that $q^8\neq 1$. Then the elements* $f_{m_1n_1}\cdots$