41. A Note on the Nash-Moser Implicit Function Theorem

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1. Problem and result. The purpose of this note is to give a simple and natural treatment of the Nash-Moser implicit function theorem established by Nash [2], Moser [1] and Schwartz [3]. (See also Sergeraert [4].) Nash's original idea (refined by Moser and Schwartz) seems to appear in the elegant form in our treatment.

Let $\{X_i\,;0\leq i<m\}$, $\{Y_i\,;0\leq i<m\}$ and $\{Z_i\,;0\leq i<m\}$, $11\leq m\leq\infty$, be three discrete Banach scales with norms $|\ \ |_i$, $\|\ \ \|_i$ and $\|\ \ \|_i$ such that $|\ \ |_i\leq|\ \ |_{i+1}$, $\|\ \ \|_i\leq\|\ \ \|_{i+1}$ and $\|\ \ \|_i\leq\|\ \ \|_{i+1}$. We assume $\{X_i\}$ is tame:

(X.0) There exists a (linear) smoothing operator $S(t)$, $t\in[1,\infty)$, from X_0 to X_{m-1} satisfying

(1) $|S(t)u|_k\leq c_0(k)t^{k-j}|u|_j,\quad u\in X_j,\quad j\leq k<m,$

(2) $|u-S(t)u|_j\leq c_0(k)t^{-(k-j)}|u|_k,\quad u\in X_k,\quad j\leq k<m.$

For a Banach space X with the norm $|\ \ |$, we put $X(r)=\{u\in X\,;|u|\leq r\}$. Let F be a continuous function from $X_1(R)\cap X_k\times Y_1(r)\cap Y_k$ into Z_{k-1} satisfying

(F.1) $\|F(u,y)\|_{k-1}\leq c_1(k)\{|u|_k+\|y\|_k\},\quad 1\leq k<m,$

(F.2) F has the Fréchet-derivative $F'_u(u,y)$ satisfying

(1) $F'_u(u,y)v$ is continuous from $X_1(R)\times Y_1(r)\times X_1$ into Z_0,

(2) $\|F'_u(u,y)v\|_0\leq c_2|v|_1$,

(3) $\|F(u+v,y)-F(u,y)-F'_u(u,y)v\|_0\leq c_2|v|_1^2$
for u and $v\in X_1(R)$ with $u+v\in X_1(R)$ and $y\in Y_1(r)$,

(F.3) $F'_u(u,y)$ has a right-inverse $L(u,y)$, $F'_u(\cdot)L(\cdot)=1$, satisfying

(1) $L(u,y)z$ is continuous from $X_1(R)\cap X_k\times Y_1(r)\cap Y_k\times Z_{k-1}$ into X_{k-1}, $1\leq k<m$,

(2) $|L(u,y)z|_0\leq c_3\|z\|_0,\quad u\in X_1(R),\quad y\in Y_1(r),\quad z\in Z_0,$
$|L(u,y)z|_{k-1}\leq c_3(k)\{(1+|u|_k+|y|_k)\|z\|_0+\|z\|_{k-1}\}$
for $u\in X_1(R)\cap X_k$, $y\in Y_1(r)\cap Y_k$ and $z\in Z_{k-1}$, $2\leq k<m$.

We consider the following equation

$$F(u,y)=0 \tag{1.1}$$

for each small $y\in Y_k$. We define a function $G(t,u,y)$ by

$$G(t,u,y)=u-S(t)L(u,y)F(u,y). \tag{1.2}$$

By Newton's method, we construct the approximate sequence $\{u_n\}$:

$$\begin{aligned} &u_0=0,\\ &u_{n+1}=G_{n+1}(y)=G(t_n,u_n,y)=u_n-S(t_n)L(u_n,y)F(u_n,y),\\ &t_n=e^{\gamma\kappa^n},\quad \gamma\geq 1,\quad 4/3\leq\kappa\leq 3/2,\quad n=0,1,2,\cdots. \end{aligned} \tag{1.3}$$

With an appropriate choice of γ and k, u_n converges to a solution u of (1.1)