12. *A Proof of Existence of The Stable Jacobi Tensor*

By Nobuhiro INNAMI
Department of Mathematics, Nagoya Institute of Technology

(Communicated by Kunihiko KODAIRA, M. J. A., Feb. 12, 1987)

0. Let M^n be a complete Riemannian manifold without conjugate points and R the curvature tensor of M. Let $\gamma:(-\infty,\infty)\to M$ be a geodesic and let $E_1,\cdots,E_n$ be a parallel orthonormal frame field along γ with $E_n(t)=\dot\gamma(t)$. We consider the $(n-1)\times(n-1)$ matrix differential equation

(J) $$D''(t)+R(t)D(t)=0,$$

where $R(t)_{ij}=\langle R(E_i,E_n)E_n,E_j\rangle(t)$ for any $t\in(-\infty,\infty)$. If $D(t)$, $-\infty<t<\infty$, is a solution of (J) and x is a parallel vector field along γ, then $D(t)x(t)$, $-\infty<t<\infty$, is a Jacobi field along γ. The following theorems play important roles in the study of manifolds without conjugate points.

Theorem 1. *Let M be a complete simply connected Riemannian manifold without conjugate points and $\gamma:(-\infty,\infty)\to M$ a geodesic. If $D_s(t)$, $-\infty<t<\infty$, are the solution of* (J) *with $D_s(0)=I$, $D_s(s)=0$ for all $s>0$, then the sequence D_s converges to a Jacobi tensor D along γ with $D(0)=I$,* $\det D(t)\neq 0$ *for any $t\in(-\infty,\infty)$ as $s\to\infty$.*

Theorem 2. *Let M and γ be as above. Then, there is a symmetric matrix field A along γ which satisfies the Ricatti equation, namely $A'(t)+A(t)^2+R(t)=0$ for any $t\in(-\infty,\infty)$.*

The theorems were originally proved by Hopf [5] and Green [4] under a more general setting. The proof was explained by Eberlein [1], Eschenburg-O'Sullivan [2] and Goto [3]. The purpose of the present note is to give a geometrical and visual proof which is simpler to some readers. The different point from their proof is that we prove Theorem 2 before Theorem 1. Theorem 1 is an immediate consequence from Theorem 2.

1. Since M is simply connected, all geodesics $\alpha:(-\infty,\infty)\to M$ are minimizing and M is diffeomorphic to E^n. In particular, all spheres are of class C^∞.

Let γ and D_s be as in Theorem 1. $D_s(t)$, $-\infty<t<\infty$, is obtained by the following way: Let $S(\gamma(0),\gamma(s))$ be the sphere with center $\gamma(s)$ through $\gamma(0)$ and let v be the unit normal vector field on $S(\gamma(0),\gamma(s))$ pointing $\gamma(s)$. We consider a map $\phi:S(\gamma(0),\gamma(s))\times(-\infty,\infty)\to M$ given by $\phi(q,t)=\exp tv(q)$. We denote by ϕ_t the map $q\to\phi(q,t)$. If $c:(-\varepsilon,\varepsilon)\to S(\gamma(0),\gamma(s))$, $c(0)=\gamma(0)$, is a curve, then $\phi\circ(c\times id):(-\varepsilon,\varepsilon)\times(-\infty,\infty)\to M$ is a geodesic variation, and, thus, $\phi_{t*}x$, $-\infty<t<\infty$, is a Jacobi field along γ for any $x\in T_{\gamma(0)}S(\gamma(0),\gamma(s))$. Hence,

$$D_s(t)=\phi_{t*}\circ P_t^{-1}$$

for any $t\in(-\infty,\infty)$, where $P_t:T_{\gamma(0)}M\to T_{\gamma(t)}M$ is the parallel translation along γ.