90. *Graphs with Given Countable Infinite Group*

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§ 1. **Introduction.** In this note we shall prove the following

Theorem. *Let Δ be any graph which is a constant link of a finite graph and which has at least one isolated vertex and at least three vertices. Then for any countable group G there are infinitely many connected graphs Γ with constant link Δ and* $\mathrm{Aut}\,\Gamma\cong G$.

Since nK_1 is the constant link of $K_{n,n}$, we have the following

Corollary. *For any countable group G and any integer $n\geqq 3$ there are infinitely many connected n-regular graphs Γ with* $\mathrm{Aut}\,\Gamma\cong G$.

The case $n=3$ and with finite group G of this corollary was proved by Frucht [1] and this result was extended to general $n\geqq 3$ by Sabidussi [2]. The case with finite group G of our theorem was proved by Vogler [4]. Our proof is an extension of [4]. We shall use the same notations as in [4].

§ 2. *Proof of Theorem.* First we refer to the following lemma without proof, whose proof is similar to that in [4 ; Theorem 1].

Lemma 1. *Let G be a countable group, Δ a constant link of a finite graph with at least three vertices and at least one isolated vertex. If for each $k=3,4,5$ there are infinitely many connected k-regular prime graphs $\prod_k$ with $\mathrm{Aut}\prod_k\cong G$ and a stable k-coloring, then there are infinitely many connected graphs Γ with constant link Δ and* $\mathrm{Aut}\,\Gamma\cong G$.

Thus it is sufficient to prove the next lemma to prove our theorem.

Lemma 2. *Let G be a countable group. Then for each $k=3,4,5$ there are infinitely many connected k-regular prime graphs $\prod_k$ with $\mathrm{Aut}\prod_k\cong G$ and a stable k-coloring.*

Proof. First we show that for each $k=3, 4, 5$ there is a connected k-regular prime graph Γ_k with $\mathrm{Aut}\,\Gamma_k\cong G$ and a stable k-coloring. If G is generated by a finite number of its elements, we see the existence of such a graph Γ_k for each $k=3,4,5$ by graphs similarly constructed to those in [1 ; Theorem 4.1], [2 ; Theorem 3.7] and [4 ; Lemma 5]. So we assume for a while that G is not generated by any finite subset. Let $S=\{x_i : i\in N\}$ be an infinite subset of G satisfying $S\not\ni 1$ and $\langle S\rangle=G$. Let us set $G_i=\langle x_1, x_2, \cdots, x_i\rangle$. Now for every integer $i\geqq 2$ if x_i is contained in G_{i-1}, we remove x_i from S. Consequently, $G=\langle S\rangle$ holds and there is no finite subset $\{s_i : i=1,2,\cdots,t\}$ of S satisfying $s_1^{\varepsilon_1}s_2^{\varepsilon_2}\cdots s_t^{\varepsilon_t}=1$ with $\varepsilon_j=\pm 1$. Hereafter we set $S=\{y_i : i\in N\}$. Let us define graphs Γ_3, Γ_4 and Γ_5 as follows:

$V(\Gamma_3)=V(\Gamma_4)=V(\Gamma_5)=\{(j, g) : j\in N,\ g\in G\}$,

$E(\Gamma_3)=\{[(1,g),(2,g)],\ [(1,g),(3,g)],\ [(1,g),(4,g)],\ [(2,g),(5,g)],$