67. On the Generators of Exponentially Bounded C-semigroups

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1. Introduction. Let X be a Banach space, and let $C: X \to X$ be an injective bounded linear operator with dense range. According to Davies and Pang [1], we say that $\{S(t): t \geqq 0\}$ is an *exponentially bounded C-semigroup* if $S(t): X \to X$, $0 \leqq t < \infty$, is a family of bounded linear operators satisfying

(1.1) $S(t+s)C = S(t)S(s)$ for $t, s \geqq 0$, and $S(0) = C$,

(1.2) for every $x \in X$, $S(t)x$ is continuous in $t \geqq 0$,

(1.3) there exist $M \geqq 0$ and real a such that $\|S(t)\| \leqq Me^{at}$ for $t \geqq 0$.

For every $t \geqq 0$, let $T(t)$ be the closed linear operator defined by

$$T(t)x = C^{-1}S(t)x \qquad \text{for } x \in D(T(t))$$

with $D(T(t)) = \{x \in X : S(t)x \in R(C)\}$. We define the operator G by

$$D(G) = \{x \in R(C) : \lim_{t \to 0+} (T(t)x - x)/t \text{ exists}\}$$

and

$$Gx = \lim_{t \to 0+} (T(t)x - x)/t \qquad \text{for } x \in D(G). \tag{1.4}$$

For every $\lambda > a$, define the bounded linear operator $L_\lambda: X \to X$ by

$$L_\lambda x = \int_0^\infty e^{-\lambda t} S(t)x \, dt \qquad \text{for } x \in X. \tag{1.5}$$

It is known that L_λ is injective and the closed linear operator Z defined by

$$Zx = L_\lambda^{-1}(\lambda L_\lambda - C)x = (\lambda - L_\lambda^{-1}C)x \qquad \text{for } x \in D(Z) \tag{1.6}$$

with $D(Z) = \{x \in X : Cx \in R(L_\lambda)\}$ is independent of $\lambda > a$. (See [1].) The operator Z will be called the *generator* of $\{S(t): t \geqq 0\}$.

Remark. If $C = I$ (the identity on X), then every exponentially bounded C-semigroup is a C_0-semigroup in the ordinary sense. In this case, (1.1) and (1.2) imply (1.3), and the generator Z coincides with G defined by (1.4) (see [2, 3]). However, these do not hold in general (see [1]).

The purpose of this paper is to prove the following theorems.

Theorem 1. *Let $\{S(t): t \geqq 0\}$ be an exponentially bounded C-semigroup satisfying $\|S(t)\| \leqq Me^{at}$, and let G be the operator defined by* (1.4). *Then G is closable and $\bar{G}$ (the closure of G) is a densely defined linear operator in X satisfying the following conditions*

(a_1) *$\lambda - \bar{G}$ is injective for $\lambda > a$,*

(a_2) *$D((\lambda - \bar{G})^{-n}) \supset R(C)$ for $n \geqq 1$ and $\lambda > a$,*

(a_3) *$\|(\lambda - \bar{G})^{-n}C\| \leqq M/(\lambda - a)^n$ for $n \geqq 1$ and $\lambda > a$,*

(a_4) *$(\lambda - \bar{G})^{-1}Cx = C(\lambda - \bar{G})^{-1}x$ for $x \in D((\lambda - \bar{G})^{-1})$ and $\lambda > a$.*

Theorem 2. *If T is a densely defined closed linear operator in X*