93. On Certain Cubic Fields. VI

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1. The notations E_F, E_F^+, $\mathcal{O}_F$ for an algebraic number field F, D_h for a polynomial $h(x)\in \boldsymbol{Z}[x]$ and $D_F(\alpha)$ for an algebraic number α in F have the same meanings as in [3].

In this note, we shall consider totally real cubic fields K with the properties:

(I) $\theta,\ \theta+1\in E_K$

(II) $\mathcal{O}_K=\boldsymbol{Z}+\boldsymbol{Z}\theta+\boldsymbol{Z}\theta^2$.

These fields will be called for convenience *primitive with two consecutive units*, in short *P-C* fields. We shall prove

Theorem. In *P-C* fields, we have $E_K=\langle\pm 1\rangle\times\langle\theta,\ \theta+1\rangle$.

2. Now we can distinguish four cases:

(1) $\theta,\ -1-\theta\in E_K^+$ (2) $\theta,\ 1+\theta\in E_K^+$

(3) $-\theta,\ -1-\theta\in E_K^+$ (4) $-\theta,\ 1+\theta\in E_K^+$

In the case (1), we have $N_{K/\boldsymbol{Q}}\theta=1$, $N_{K/\boldsymbol{Q}}(1+\theta)=-1$ which implies $\mathrm{Irr}(\theta;\boldsymbol{Q})=x^3-mx^2-(m+3)x-1$, $m\in\boldsymbol{Z}$, and in the case (2), we have $N_{K/\boldsymbol{Q}}\theta=1$, $N_{K/\boldsymbol{Q}}(1+\theta)=1$ which implies $\mathrm{Irr}(\theta;\boldsymbol{Q})=x^3-nx^2-(n+1)x-1$, $n\in\boldsymbol{Z}$. The cases (3), (4) can be reduced to the case (2) by replacing θ respectively by $-1-\theta$ and $-(1+\theta)^{-1}$. Accordingly, we have to consider two kinds of fields (*P-C*1) and (*P-C*2), which are *P-C* fields with properties (1) respectively (2).

Now we have

Theorem 1. *Cubic field* $K=\boldsymbol{Q}(\theta)$ *with* $\mathrm{Irr}(\theta;\boldsymbol{Q})=f(x)\in\boldsymbol{Z}[x]$ *is* (*P-C*1) *field, if and only if* $f(x)=x^3-mx^2-(m+3)x-1$, $m\in\boldsymbol{Z}$ *and* $\sqrt{D_f}=m^2+3m+9$ *is square free.*

In fact, (1) is equivalent with $\mathrm{Irr}(\theta;\boldsymbol{Q})=f(x)=x^3-mx^2-(m+3)x-1$ and in this case K is Galois and so totally real, and (II) holds if and only if $\sqrt{D_f}$ is square free.

Theorem 2. *Cubic field* $K=\boldsymbol{Q}(\theta)$ *with* $\mathrm{Irr}(\theta;\boldsymbol{Q})=g(x)\in\boldsymbol{Z}[x]$ *is* (*P-C*2) *field, if and only if* $g(x)=x^3-nx^2-(n+1)x-1$, $n\in\boldsymbol{Z}$, $D_g=(n^2+n-3)^2-32>0$ *is square free.*

In fact, (2) is equivalent with $\mathrm{Irr}(\theta;\boldsymbol{Q})=x^3-nx^2-(n+1)x-1$ and $D_g>0$ means that K is totally real, and (II) means that D_g is square free.

3. *Proof of Theorem.* We shall prove this theorem in two cases: (*P-C*1) fields and (*P-C*2) fields.