7. Zeros, Eigenvalues and Arithmetic

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Let γ run over the imaginary parts of the non-trivial zeros of the Riemann zeta function $\zeta(s)$. Let $1/4+r^2$ run over the eigenvalues of the discrete spectrum of the Laplace-Beltrami operator in L^2 (the upper half plane $/\Gamma$), where we take $\Gamma=PSL(2,Z)$. Let α be a positive number. Here we introduce the zeta functions defined by

$$Z_\alpha(s)=\sum_{\gamma>0}\frac{\sin(\alpha\gamma)}{\gamma^s} \quad\text{and}\quad \mathfrak{Z}_\alpha(s)=\sum_{r>0}\frac{\sin(\alpha r)}{r^s}.$$

We are concerned with their analytic properties and their arithmetic.

To state our results we shall introduce some notations. $\Lambda(\cdot)$ is the von Mangoldt function. Let $\{P_0\}$ run over the primitive hyperbolic conjugacy classes in $PSL(2,Z)$. $N(P_0)$ denotes the square of the eigenvalue (>1) of a representative P_0. For a hyperbolic conjugacy class $\{P\}$ satisfying $P=P_0^k$ with a natural number k, we put $\tilde{\Lambda}(P)=(\log N(P_0))/(1-N(P)^{-1})$, where $N(P)=N(P_0)^k$. $A(\Gamma)$ denotes the area of the fundamental domain of Γ, which is equal to $\pi/3$. We assume the Riemann Hypothesis to get the results on γ or on $Z_\alpha(s)$. The following theorem describes a property of the distribution of γ or r.

Theorem 1. *Let $T>T_0$ and α be a positive number. Then*

$$\text{i)}\quad \sum_{0<\gamma\leqq T} e^{i\alpha\gamma}=-\frac{1}{2\pi}\frac{\Lambda(e^\alpha)}{e^{\alpha/2}}T+\frac{e^{i\alpha T}}{2\pi i\alpha}\log T+O\left(\frac{\log T}{\log\log T}\right)$$

and

$$\text{ii)}\quad \sum_{0<r\leqq T} e^{i\alpha r}=\frac{1}{\pi}\frac{\Lambda(e^{\alpha/2})}{e^{\alpha/2}}T+\frac{A(\Gamma)}{2\pi i\alpha}Te^{i\alpha T}+\frac{e^{-\alpha/2}}{2\pi}\left(\sum_{N(P)=e^\alpha}\tilde{\Lambda}(P)\right)T+O\left(\frac{T}{\log T}\right).$$

We remark that i) is a refinement of Landau's theorem and has been proved by the author in [3]. ii) can be proved by the same method. Venkov [11] has studied the asymptotic behavior of the sum $\sum_{r>0}\cos(\alpha r)e^{-tr^2}$ as $t\to+0$. We see by this theorem that for any positive α as $m\to\infty$, $\sum_{0<\gamma\leqq m}\sin(\alpha\gamma)/\gamma^s$ converges to $Z_\alpha(s)$ if $\mathrm{Re}\,s>0$ and $\sum_{0<r\leqq m}\sin(\alpha r)/r^s$ converges to $\mathfrak{Z}_\alpha(s)$ if $\mathrm{Re}\,s>1$. Using the Poisson summation formula and the Selberg trace formula, we can show the following theorem.

Theorem 2. *For any positive α, $Z_\alpha(s)$ and $\mathfrak{Z}_\alpha(s)$ are entire.*