49. On Obstructions of Infinitesimal Lifting

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Let X be the analytic submanifold of an algebraic manifold P defined by an ideal sheaf I_X. Let E be a holomorphic vector bundle on P and $E|_X$ its restriction to X. Then, we consider the formal completion $\hat{P}$ of P and the formal completion $\hat{E}$ of E along X.

Assume that a holomorphic section $\bar{\sigma}$ of $E|_X$ is given. Then, it is not easy to find a criterion in terms of $\bar{\sigma}$ and $E|_X$ for liftability of the section $\bar{\sigma}$ to a section of $\hat{E}$. Getting a lifting $\hat{\sigma}$ in $\hat{E}$ of $\bar{\sigma}$ is equivalent to constructing a system $\{\sigma_{(\mu)}\}_{\mu=1}^{\infty}$ of sections of $\{O_P(E)\otimes O_P/I_X^{\mu}\}$ such that (1) $\sigma_{(1)}=\bar{\sigma}$, (2) $\pi_{\mu+1}(\sigma_{(\mu+1)})=\sigma_{(\mu)}$ for every $\mu\geqq 1$ where $\pi_{\mu+1}$; $O_P(E)\otimes O_P/I_X^{\mu+1}\to O_P(E)\otimes O_P/I_X^{\mu}$ is a natural projection.

When $\sigma_{(\mu)}$ is given, the obstruction for finding a lifting $\sigma_{(\mu+1)}$ of $\sigma_{(\mu)}$ is $\delta_\mu(\sigma_{(\mu)})$ which appears in the following sequence,

$$0\longrightarrow H^0(O_X(S^{(\mu)}(N^*_{X/P})\otimes E|_X))\longrightarrow H^0(O_P(E)\otimes O_P/I_X^{\mu+1})\longrightarrow H^0(O_P(E)\otimes O_P/I_X^{\mu})\xrightarrow{\delta_\mu}$$
$$H^1(O_X(S^{(\mu)}(N^*_{X/P})\otimes E|_X))\longrightarrow\cdots$$

where $N^*_{X/P}$ is the conormal bundle of X in P. $\delta_\mu(\sigma_{(\mu)})$ is called "μ-th obstruction". The following fact should be pointed out here: Even if $\delta_{(\mu_0)}(\sigma_{(\mu_0)})$ does not vanish for some system $\{\sigma_{(\mu)}\}_{\mu=1}^{\mu_0}$ in a course of constructing a system, this does not mean the impossibility of lifting of $\bar{\sigma}$. We may find another system $\{\tau_{(\mu)}\}_{\mu=1}^{\mu_0}$ with $\delta_{(\mu_0)}(\tau_{(\mu_0)})=0$ which extends $\bar{\sigma}$.

Now, let us consider a following special case which is useful in the study of the defining equations. Let P be a projective space $\boldsymbol{P}^N(\boldsymbol{C})$ and E be $\Omega_P^1(\nu)=\Omega_P^1\otimes O_P(1)^{\otimes\nu}$.

Theorem. *Assume the embedding of X into $P=\boldsymbol{P}^N$ is projectively normal. Then the given section $\bar{\sigma}$ of $H^0(X, \Omega_P^1(\nu)\otimes O_X)$ can be lifted to a section of $H^0(\hat{P}, \hat{\Omega}_P^1(\nu))$ if and only if the first obstruction $\delta_1(\bar{\sigma})$ vanishes.*

The full proof will be given in [1]. Here, we shall give only a rough sketch of the proof. First, we make a special vector subspace V_0 of $H^0(X, \Omega_P^1(\nu)\otimes O_X)$ and another operator

$$\mathfrak{D};\ H^0(X, \Omega_P^1(\nu)\otimes O_X)\longrightarrow H^1(X, O_X(N^*_{X/P})\otimes\Omega_P^1(\nu))$$

by using special property of $\Omega_P^1(\nu)$. And then, we show that on V_0, $\mathfrak{D}$