21. C_l-Metrics on Spheres

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1. Let (M, g) be a riemannian manifold. Then we call g a C_l-metric if all of its geodesics are closed and have the common length l. As is well-known, the standard metric on the unit sphere S^n is a $C_{2\pi}$-metric. Suppose $\{g_t\}$ is a one-parameter family of $C_{2\pi}$-metrics on S^n such that g_0 is the standard one. Put

$$\frac{d}{dt} g_t|_{t=0} = h.$$

We call such a symmetric 2-form h an infinitesimal deformation. It is known that each infinitesimal deformation h satisfies

$$(*) \qquad \int_0^{2\pi} h(\dot{\gamma}(s), \dot{\gamma}(s))ds = 0$$

for any geodesic $\gamma(s)$ of (S^n, g_0) parametrized by arc-length (cf. [1] p. 151). V. Guillemin has proved in [2] that in the case of S^2 the condition $(*)$ is also sufficient for a symmetric 2-form h to be an infinitesimal deformation.

The purpose of this note is to show that the situation is completely different in the case of S^n $(n \geqq 3)$. We shall give another necessary condition for a symmetric 2-form h to be an infinitesimal deformation (Theorem 1). And we shall give a partial result for what h satisfies this condition (Propositions 2, 3).

2. We denote by $\mathcal{K}_2$ the vector space of symmetric 2-forms on S^n which satisfy $(*)$. Let $\sharp : T^*S^n \to TS^n$ be the bundle isomorphism defined by

$$g_0(\sharp(\lambda), v) = \lambda(v), \quad \lambda \in T_x^*S^n, \quad v \in T_xS^n, \quad x \in S^n.$$

Let E_0 be the function on T^*S^n such that

$$E_0(\lambda) = \frac{1}{2} g_0(\sharp(\lambda), \sharp(\lambda)), \qquad \lambda \in T^*S^n.$$

Consider the usual symplectic structure on T^*S^n, and let X_{E_0} be the symplectic vector field on T^*S^n defined by the hamiltonian E_0. E_0 and X_{E_0} are called the energy function and the geodesic flow associated with the metric g_0 respectively. We denote by $\{\xi_t\}$ the one-parameter group of transformations of T^*S^n generated by X_{E_0}. Then $\{\xi_t\}$ induces a free S^1-action of period 2π on the unit cotangent bundle S^*S^n. We define an operator $G : C^\infty(S^*S^n) \to C^\infty(S^*S^n)$ by