117. *Product Formula for Nonlinear Semigroups in Hilbert Spaces*

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1. Introduction. Let H be a real Hilbert space. Let A and B be maximal monotone (multi-valued) operators in H such that $A+B$ is also maximal monotone in H. (We refer to the work of Brezis [2] for basic results concerning maximal monotone operators.) Let $\{S_A(t); t\geqq 0\}$, $\{S_B(t); t\geqq 0\}$ and $\{S_{A+B}(t); t\geqq 0\}$ be the contractive semigroups in H generated by $-A$, $-B$ and $-(A+B)$, respectively. The purpose of this paper is to show the following result.

Theorem. *If there exists a closed convex set $C\subset\overline{D(A)\cap D(B)}$ such that $(I+\lambda A)^{-1}(C)\subset C$ and $(I+\lambda B)^{-1}(C)\subset C$ for $\lambda>0$, then*

(1.1) $$S_{A+B}(t)x=\lim_{n\to\infty}(S_A(t/n)S_B(t/n))^n x$$

for each $x\in C\cap\overline{D(A)\cap D(B)}$ and each $t\geqq 0$ and the convergence is uniform on each finite interval of $[0,\infty)$.

This theorem was proved by Brezis and Pazy in [3] with the extra assumption that A and B are single-valued. Similar results are obtained for some Banach spaces as well and will be treated in the forthcoming paper [5] of the author.

2. Proof of the theorem. (Step 1.) By Proposition 4.5 in [2], $S_A(t)$ and $S_B(t)$ are contractions on C into itself. So we shall prove the convergence

$$\lim_{t\to 0+}(I+\lambda t^{-1}(I-S_A(t)S_B(t)))^{-1}x=(I+\lambda(A+B))^{-1}x$$

for each $x\in C\cap\overline{D(A)\cap D(B)}$ and each $\lambda>0$, from which our assertion is derived through Theorem 4.3 of [2]. To this end, let $\lambda>0$, fix any $x\in C\cap\overline{D(A)\cap D(B)}$ and set $u(t)$ any y_0 to be $(I+\lambda t^{-1}(I-S_A(t)S_B(t))^{-1}x$ and $(I+\lambda(A+B))^{-1}x$, respectively. It can easily be seen that

(2.1) $$\lambda^{-1}(u(t)-x)=t^{-1}(S_A(t)S_B(t)u(t)-u(t)),$$

$u(t)$ are contained in C for all $t>0$ and $u(t)$ is bounded as $t\to 0+$. Since $S_A(t)$ and $S_B(t)$ are contractions from C into itself, the indefinite integrals

$$v(t)=t^{-1}\int_0^t S_B(s)u(t)ds \quad\text{and}\quad w(t)=t^{-1}\int_0^t S_A(s)S_B(t)u(t)ds$$

are contained in C for all $t>0$ and bounded as $t\to 0+$. Therefore, one can choose a null sequence $\{t_n\}$ of positive numbers such that

(2.2) $$u(t_n)\rightharpoonup u_0,\ v(t_n)\rightharpoonup v_0 \quad\text{and}\quad w(t_n)\rightharpoonup w_0$$

as $n\to\infty$, where the symbol $\rightharpoonup$ means the weak convergence and u_0, v_0