113. On Certain Diophantine Equations in Algebraic Number Fields

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1. Diophantine equations of the following type have been discussed by many authors.

Let K be an algebraic number field of some type (e.g. totally real, abelian over $\boldsymbol{Q}$, or "almost real" cf. [3]), α, β given roots of unity, and m a given natural number. Find the solutions of the equation:

(1) $$\xi^m+\alpha=\eta, \quad \xi \in K(\beta+\beta^{-1}), \quad \eta \in U_{K(\beta)},$$

where U_F will mean the group of units of the algebraic number field F. (Cf. [1]–[5]. E.g. it is shown in [3] that when K is almost real, $\alpha=\beta=-1$, $m\geqq 3$, $\xi \in U_K$, then the only possible solutions are given by ξ $=$a root of unity. This covers the results of [2], [5].)

We shall denote in the following the ring of integers of the field F by $\mathcal{O}_F$. p will mean an odd prime, and for any natural number n, ζ_n will mean a primitive n th root of unity.

Remark. From (1) follows immediately $\xi \in \mathcal{O}_{K(\beta+\beta^{-1})}$.

In this note, we prove the following three theorems:

Theorem A. *Suppose K to be totally real and $m=1$ in* (1).

(I) *If $\alpha=\beta=\zeta_4$, then $\xi=0$.*

(II) *If $\alpha=\beta=\zeta_p$, then $\xi=(\zeta_p^{c-1}-\zeta_p)/(1-\zeta_p^c)$ with $c \in \{1,2,\cdots,p-1\}$.*

(III) *If $\alpha=\beta=\zeta_p$, K is moreover non-abelian and of prime degree over $\boldsymbol{Q}$, then $\xi=0$ or 1.*

Remark. To Theorem A may be associated a problem posed by Julia Robinson, cited in [4], asking for possibilities of expressing 1 as the difference of two units in an algebraic number field.

Theorem B. *Suppose K to be totally real, $m\geqq 2$, $\alpha=\beta=1$, $\eta\neq 1$. Then the only possible solutions of* (1) *are given by $\xi=$a root of unity.*

Theorem C. *Suppose $K/\boldsymbol{Q}$ to be abelian, $m=2$, $\alpha=1$ and $\beta=\zeta_{4k}$ where k is an odd natural number $\geqq 3$. Then the only solution of* (1) *is $\xi=0$, $\eta=1$.*

2. *Proof of Theorem A.* Our equation is in this case $\xi+\alpha=\eta$, $\alpha=\beta=\zeta_4$ or ζ_p, $\xi \in K(\alpha+\alpha^{-1})$, $\eta \in U_{K(\alpha)}$. Notice first ξ should be $\in \mathcal{O}_{K(\alpha)}$ as α, $\eta \in \mathcal{O}_{K(\alpha)}$.

(I) Suppose $\xi\neq 0$. As $K(\zeta_4+\zeta_4^{-1})=K$ is totally real, all conjugates ξ' of ξ are real, and $|\xi'\pm\xi_4|>1$, so that $\xi+\zeta_4$ can not be $\in U_{K(\zeta_4)}$.