111. A Note on the Convolution Theorem on Functions of Several Variables

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1. J. Mikusiński and C. Ryll-Nardzewski [1], [2] generalized the well-known Titchmarsh convolution theorem, which plays a fundamental roll in Mikusiński's operational calculus [3], to functions of several variables; see K. Yosida and S. Okamoto [4] for their operational calculus without appealing to the Titchmarsh convolution theorem. Mikusiński and Ryll-Nardzewski showed:

Theorem. *Let* $T>0$, *and* $\omega\in\boldsymbol{R}_+^n$; $\boldsymbol{R}_+^n=\{x=(x_1,\cdots,x_n);\, x_j>0\ (j=1,\cdots,n)\}$. *Let* f, g *be two integrable functions on* S_T; $S_T=\{x\in\boldsymbol{R}_+^n;\, 0<x\cdot\omega<T\}$, $(x\cdot\omega=\sum_{j=1}^n x_j\omega_j)$. *If the convolution of* f *and* g *vanishes almost everywhere* (a.e.) *in* S_T:

$$(1)\qquad f*g(x)=\int_0^{x_1}\cdots\int_0^{x_n} f(x-y)g(y)dy_1\cdots dy_n=0,\ \text{a.e. in } S_T$$

$(x=(x_1,\cdots,x_n);\, y=(y_1,\cdots,y_n))$, *then there are two non-negative numbers* T_1, T_2 *with* $T_1+T_2\geq T$ *such that* $f(x)=0$, a.e. *in* S_{T_1} *and* $g(x)=0$, a.e. *in* S_{T_2}.

This theorem was first formulated and proved by E. Titchmarsh [5], [6] for $n=1$. The original proof was based on a difficult theorem in the complex analytic function theory; see M. Crum [7] and J. Dufresnoy [8] for simpler proofs.

Elementary proofs, based exclusively on the method of functions of a real variable, were given by Mikusiński [9], and Mikusiński and Ryll-Nardzewski [3] by making use of the Lerch moment theorem. Mikusiński and Ryll-Nardzewski extended the Titchmarsh convolution theorem to functions of several variables by a geometrical method. J. Lions [10], [11] obtained similar results by making use of the Fourier transform, assuming that the functions considered satisfy some growth conditions of exponential type. The purpose of the present note is to show the Titchmarsh-Mikusiński-Ryll·Nardzewski theorem by making use of the Fourier transform without imposing any such restriction on functions. Prof. K. Yosida kindly communicated to the author another proof ([12]) which makes use of the change of variables in the double integration, the Liouville theorem and the Weierstrass polynomial approximation theorem when the present work was almost completed.