99. *Characterization of Extremely Amenable Semigroups with a Unique Invariant Mean*

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§ 1. **Introduction.** Let S be an abstract semigroup and $m(S)$ the Banach space of all bounded real functions on S with the supremum norm. Let ψ be a mean on S i.e., ψ is a linear functional on $m(S)$ such that $\|\psi\|=\psi(I_S)=1$, where in general I_A is the characteristic function of any subset A of S. ψ is called *multiplicative* if $\psi(fg)=\psi(f)\psi(g)$ for any $f, g \in m(S)$. ψ is said to be *left* [*right*] *invariant* if $\psi(f)=\psi({}_sf)[\psi(f)=\psi(f_s)]$ for any $f \in m(S)$ and $s \in S$, where ${}_sf(t)=f(st)$ and $f_s(t)=f(ts)$ $(t \in S)$. By $LM(S)$ [$RM(S)$] we denote the set of all left [right] invariant means on S, and by $MLM(S)$ [$MRM(S)$] denote the set of all multiplicative left [right] invariant means on S. Any element in $LM(S) \cap RM(S)$ is called an *invariant mean* on S. We say that S is *left* [*right*] *amenable* if $LM(S)$ [$RM(S)$] $\neq \phi$ (empty). In particular S is called *extremely left* [*right*] *amenable* if $MLM(S)$ [$MRM(S)$] $\neq \phi$. Further S is called *extremely amenable* if $MLM(S) \cap MRM(S) \neq \phi$. It is proved in Granirer [2] that S is extremely left amenable if and only if it has the following property:

(CRZ) *For any $a, b \in S$ there exists $c \in S$ such that $ac=bc=c$.*
Suppose now S is extremely left amenable. Let us introduce a pseudo-order relation "$\leq$" in S defined as follows: For any $a, b \in S$, $a \leq b$ if either $a=b$ or $ab=b$. Then by virtue of (CRZ), $(S, \leq)$ is regarded as a directed set. A subset A of S is called *cofinal* (with respect to $\leq$) if for any given $s \in S$ there exists $t \in A$ such that $s \leq t$. For any fixed $p \in S$ we denote by δ_p a mean on S defined by $\delta_p(f)=f(p)$ $(f \in m(S))$.

The purpose of this paper is to prove the following theorems concerning semigroups with a unique multiplicative invariant mean.

Theorem 1. *Let S be extremely left amenable. The following conditions for S are equivalent:*

(1) *S admits a unique multiplicative left invariant mean.*

(2) *S admits a unique left invariant mean.*

(3) *For any subset A of S there exists $t \in S$ such that either $tS \subseteq A$ or $tS \cap A=\phi$.*

(4) *S has the property that if a subset A of S satisfies $sS \cap A \neq \phi$ for all $s \in S$, then $tS \subseteq A$ for some $t \in S$.*