89. *On a Certain Property of Profinite Groups*

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1. We are led to consider a certain property of profinite groups in investigating a problem posed by Jehne [1] on Kronecker sets of algebraic number fields. Let Q be the rational number field, k a finite algebraic extension of Q and K a finite algebraic extension of k. For the extension K/k, we consider the set $P(K/k)$ of all prime divisors of k having a prime divisor of first relative degree in K/k. We call this set $P(K/k)$ the Kronecker set of K/k. For Kronecker sets, we denote equality of sets up to finite set by $\doteqdot$. Now, the problem of Jehne asks whether there exists a sequence $\{k_n\}_{n=1}^{\infty}$ of finite algebraic extensions of k such that $k_{n+1} \supsetneqq k_n$ and that $P(k_1/k) \doteqdot P(k_n/k)$ for any positive integer n. We call the above sequence $\{k_n\}_{n=1}^{\infty}$ an infinite Kronecker tower of k. In Satz 6 of [2], Klingen claimed that there exists no infinite Kronecker tower of k. As we shall see in the following, the proof of the theorem contains an argument, which is not correct. The following property of Kronecker sets is well-known:

Proposition 1. *Let k be a finite algebraic extension of Q, L a (finite or infinite) Galois extension of k and G the Galois group of L over k. Let H and H' be open subgroups of G. Let K and K' be subfields of L corresponding to the subgroups H and H' of G, respectively. Then the following conditions are equivalent:*

(1) $P(K/k) \doteqdot P(K'/k)$.

(2) $\bigcup_{g\in G} g^{-1}Hg = \bigcup_{g\in G} g^{-1}H'g$.

We owe the following lemma essentially to Klingen [2]:

Lemma 1. *Let L be a (finite or infinite) Galois extension of k and G the Galois group $G(L/k)$ of L over k. For any positive integer n, we denote by k_n a finite algebraic extension of k such that L contains k_n. We suppose that k_{n+1} contains k_n for any positive integer n. Let $K = \bigcup_{n=1}^{\infty} k_n$, let $H = G(L/K)$ and let $H_n = G(L/k_n)$. Then the following conditions are equivalent:*

(1) $P(k_1/k) \doteqdot P(k_n/k)$ *for any positive integer n.*

(2) $\bigcup_{g\in G} g^{-1}H_1 g = \bigcup_{g\in G} g^{-1}Hg$.

The following lemma follows immediately from the fact that G is a profinite group:

Lemma 2. *Let L be a (finite or infinite) Galois extension of k and K an intermediate field L over k. Let $G = G(L/k)$ and $H = G(L/K)$.*