21. On the Convergence in L of Some Special Fourier Series

By Takeshi KANO and Saburô UCHIYAMA
Department of Mathematics, Okayama University, Okayama

(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1977)

It is well known that if a real function $f=f(x)$ belongs to the class $L^p=L^p(0,2\pi)$, $p>1$, then so does also its conjugate function $\tilde{f}$, and both of their Fourier series $\boldsymbol{S}(f)$ and $\boldsymbol{S}(\tilde{f})$ converge in (the metric of) L^p. Similarly, if $f\in L=L^1$ and if $0<p<1$, then we have $f\in L^p$ and $\tilde{f}\in L^p$, and their Fourier series $\boldsymbol{S}(f)$ and $\boldsymbol{S}(\tilde{f})$ again converge in L^p, though in this case the class L^p is not a metric space. When $p=1$, however, the situation becomes different, as was first noticed by F. Riesz (cf. e.g. [1-II; Chap. VIII, § 22]) who gave a counterexample showing that in the metric space L we cannot expect a corresponding result to hold true.[1)] Actually, he constructed a function $f\in L$ and its conjugate $\tilde{f}\in L$ such that both $\boldsymbol{S}(f)$ and $\boldsymbol{S}(\tilde{f})$ unboundedly diverge in L, that is,

$$\limsup_{n\to\infty}\|S_n\|=\limsup_{n\to\infty}\|\tilde{S}_n\|=+\infty$$

which implies that

$$\limsup_{n\to\infty}\|f-S_n\|=\limsup_{n\to\infty}\|\tilde{f}-\tilde{S}_n\|=+\infty,$$

where $S_n=S_n(x)$ and $\tilde{S}_n=\tilde{S}_n(x)$ denote the n-th partial sums of $\boldsymbol{S}(f)$ and $\boldsymbol{S}(\tilde{f})$, respectively, and where $\|\ \|$ designates the ordinary L^1-norm (over the interval $(0,2\pi)$).

In the present article we shall be concerned with the problem of the convergence and divergence in L of Fourier series of sines and of cosines, with quasi-convex coefficients. Our primary aim is to prove Theorem 1 below, the proof itself of which is entirely of elementary and constructive character.

1. Theorems. Let (c_n) be an infinite sequence of real numbers. The sequence (c_n) is said to be of bounded variation, if it satisfies the condition $\sum_{n=1}^{\infty}|\Delta c_n|<+\infty$, where $\Delta c_n=c_n-c_{n+1}$, and (c_n) is quasi-convex, if $\sum_{n=1}^{\infty}n|\Delta^2 c_n|<+\infty$, where $\Delta^2 c_n=\Delta c_n-\Delta c_{n+1}$; a bounded, quasi-convex sequence (c_n) is of bounded variation.

Theorem 1. *We can find an infinite, quasi-convex null sequence (c_n) of non-negative real numbers such that the series*

$$\sum_{n=1}^{\infty} c_n \sin nx \tag{1}$$

1) This notwithstanding, it is true that if a trigonometric series converges in L to a function $f\in L$, then it is the Fourier series of the function f (cf. [1-I; Chap. I, §12]).