5. On the Nilpotency Indices of the Radicals of Group Algebras of p-Solvable Groups

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Let K be an algebraically closed field with characteristic $p>0$, G a finite group of order $p^m g'$, $(p, g')=1$, KG a group algebra of G over K, $J(KG)$ the radical of KG and $t(G)$ the nilpotency index of $J(KG)$.

For a block B of KG denote by $t(B)$ the nilpotency index of the radical $J(B)$ of B. G. O. Michler [6] showed that if a defect group D of B is cyclic and normal in G, then B is a serial ring and $t(B)=|D|$. In this paper we shall prove that when D is cyclic, B is serial if and only if $t(B)=|D|$.

D. S. Passman [9], Y. Tsushima [11] and D. A. R. Wallace [12] showed that $m(p-1)+1\leqq t(G)\leqq p^m$ provided G is p-solvable. Recently K. Motose and Y. Ninomiya [8] proved that for a p-solvable group G of p-length 1, $t(G)=p^m$ if and only if a p-Sylow subgroup P of G is cyclic. We shall generalize this result as follows: For an arbitrary p-solvable group G, $t(G)=p^m$ if and only if P is cyclic. This is an affirmative answer to Ninomiya's conjecture announced in the Summer Algebra Symposium at Matsuyama in Japan (1974).

We call a module *uniserial* if it has a unique composition series of finite length. To being with we shall prove

Proposition 1. *Let B be a block of KG with a defect group D. If D is cyclic, then $t(B)\leqq|D|$.*

Proof. We can assume that $J(B)\neq 0$. Put that $B=\sum_{i=1}^{n}\sum_{j=1}^{f_i}\oplus KGe_{ij}$, where $\{e_{ij}\}$ are orthogonal primitive idempotents of KG such that $KGe_{i1}\cong KGe_{ij}$ for $j=1,\cdots,f_i$; $i=1,\cdots,n$ and $KGe_{i1}\not\cong KGe_{k1}$ if $i\neq k$, and $e_{i1}=e_i$ for $i=1,\cdots,n$. Let $C=(c_{ik})_{1\leqq i,k\leqq n}$ be the Cartan matrix for B and t_i the least positive integer such that $J(KG)^{t_i}e_i=0$ for $i=1,\cdots,n$. Then $t(B)\leqq\max\{t_k\,|\,1\leqq k\leqq n\}=t_i$ for some i and $t_i\leqq s_i$, where $s_i=\sum_{k=1}^{n}c_{ik}$. By [4, Satz 1], there is a pair of uniserial left KG-modules L_{i1}, L_{i2} such that $J(KG)e_i=L_{i1}+L_{i2}$, $L_{i1}\cap L_{i2}\cong KGe_i/J(KG)e_i$, L_{i1} and L_{i2} have no common composition factors except $KGe_i/J(KG)e_i$, and all composition factors of L_{i1} are nonisomorphic. Again, by [4, Satz 1], $s_i=r_{i1}+(c_{ii}-1)r_{i2}$, where r_{iv} is the number of nonisomorphic composition factors of L_{iv} for $v=1,2$, and $r_{i1}+r_{i2}\leqq n+1$. If we put that $c=\max\{c_{kk}-1\,|\,1\leqq k\leqq n\}$, by [1, Theorem 1], $|D|=cn+1$. Therefore $t(B)\leqq|D|$.