88. *On the Asymptotic Distribution of the Sum of Independent Random Variables.*

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§1. Let $\{X_i\}$ $i=1, 2, \dots.$ be a sequence of independent random variables defined in a probability space (Ω, F, P). The so-called central limit theorem[1] states that when a sequence $\{X_i\}$ satisfies certain conditions then

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i(\omega) \leqq a\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-u^2/2}\,du = G(a)\,, \qquad \text{(I)}$$

where $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i$ denotes suitably normalized variable. Concerning this theorem we consider following two generalizations:

1° Replace a constant upper limit a of summation by a measurable function $g(\omega)$ defined in Ω.

2° Replace the number n of random variables of summation by a random function $N_n(\omega)$ defined in Ω.

On these generalizations many theorems have been proved[2]. Let $\{X_i\}$ be a sequence of independent random variables satisfying the central limit theorem (I). For any real numbers a and b, we define the sets $E^i_{a,b} = [\omega\,;\, a \leqq X_i(\omega) < b]$ and denote by $\bar{F}$ the smallest Borel field which includes all the sets $E^i_{a,b}$ defined for any a, b and $i=1, 2, \dots.$ We complete $\bar{F}$ with respect to the measure P and denote it by $\bar{\bar{F}}$. In §3 we prove the following:

Theorem 1. If $E \in \bar{\bar{F}}$, then

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i(\omega) \leqq a,\ E\right) = P(E)G(a)\,.$$

In order to prove this theorem we show some lemmas in §2, and in §4 we consider the above generalizations by using Theorem I.

To define and to discuss the problems on $\{X_i\}$, it is sufficient to consider the probability space (Ω, F, P) as $(\Omega, \bar{\bar{F}}, P)$. So the theorems proved in §4 give the answer of the above generalizations for independent sequence.

§2. First of all we consider a sequence $\{X_i\}$ which satisfies following conditions:

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1) H. Cranier, Random variable and its probability distribution. Cambridge (1937).

2) J. C. Smith, On the asymptotic distribution of the sum of Rademacher functions. Bull. Amer. Math. Soc., vol. 51 (1945).

H. Robbins, On the sum of random number of random variables. Bull. Amer. Math. Soc., vol. 54 (1948).

S. Takahashi, On the central limit theorem (under the press).