46. *On a Fundamental Lemma on Weakly Normal Rings*

By Tadasi NAKAYAMA

Mathematical Institute, Nagoya University

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Let R be an (associative) ring. With a subset X of R we denote by X_r (resp. X_l) the set of right (resp. left) multiplications of the elements of X onto R. The commuter $V_{\mathfrak{A}}(X_r)$ of X_r in the absolute (module-) endomorphism ring $\mathfrak{A}$ of R is nothing but the X-right-endomorphism ring of R. Now, if S is a subring of R and if the S_r-endomorphism ring $V_{\mathfrak{A}}(S_r)$ of R (which certainly contains R_l) is generated over R_l by a family of R_l-semilinear endomorphisms of R, then we say that S is a weakly normal *) subring of R. Recently the writer studied the case where the ring R and its weakly normal subring S are simple rings with minimum condition (or complete primitive rings **)) and showed that then R is fully reducible as an R_l S_r-module[8]; this enabled the writer to obtain a theorem of extension of isomorphisms of certain weakly normal subrings, which forms a generalization and a refinement of the theorems of Artin-Whaples[1] and Cartan-Dieudonné[4], to establish a simple ring generalization of the Cartan-Jacobson[3)6] Galois theory (for sfields), and further, to extend Hochschild's[5] cohomology theory of simple algebras to simple rings[8)9)10]. The purpose of the present short note is to observe that this fundamental lemma remains true also in case the subring S is not necessarily simple (or complete primitive) but merely semisimple. This extension entails a corresponding generalization in cohomology theory and has some bearings for Galois theory, though we shall not discuss these in the present note.

We prove thus

Theorem 1 (*Fundamental lemma*). *Let R be a simple ring having unit element* 1 *and satisfying minimum condition. Let S be a weakly normal semisimple subring of R containing* 1 *and satisfying minimum condition. Then R is fully reducible as an R-left- and S-right-module.*

Proof. Evidently R is S_r-fully reducible. Let

$$R = \mathfrak{N}_1 \oplus \mathfrak{N}_2 \oplus \cdots \oplus \mathfrak{N}_s$$

be the idealistic decomposition of the S_r-module R; thus each $\mathfrak{N}_i$ is homogeneously fully reducible with respect to S_r, and distinct $\mathfrak{N}_i$, $\mathfrak{N}_j$ have no mutually isomorphic minimal S_r-submodules. The S_r-endo-

*) Dieudonné 4), Nakayama 8)9)10).

**) With certain modification of definition and under certain restrictions.