# *223. On Imbeddings and Colorings of Graphs. I*

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§ 1. **Introduction.** A graph $G$ is an ordered pair $(G^0, G^1)$, where $G^0$ is a nonempty finite set of objects and $G^1$ is a set of unordered finite pairs of the elements of $G^0$, where $G^1$ can contain some pairs of the same elements of $G^0$. The objects in $G^0$ and $G^1$ are called *vertices* and *arcs* of the graph $G$, respectively.

For two graphs $G=(G^0, G^1)$ and $H=(H^0, H^1)$, $H$ is called a subgraph of $G$ and noted by $G\supset H$, if $G^0\supset H^0$ and $G^1\supset H^1$.

We can realize any graph by an 1-dimensional complex, where we assume that an arc is an open 1-simplex in the complex. A graph is *connected* when it is connected as a complex. In this paper, a graph implies a connected graph.

A subset $\delta$ of $G^0$ is called *SC-set* (same-colorable set), if for any two vertices $A$ and $B$ in $\delta$, the pair $(A, B)$ is not contained in $G^1$.

A graph $G=(G^0, G^1)$ is *n-colorable* if such $n$ SC-sets $\delta_1, \cdots, \delta_n$ exist as $G^0=\delta_1\cup\cdots\cup\delta_n$, $\delta_i\neq\phi$ and if $i\neq j$ $\delta_i\neq\delta_j$ for $i, j=1, \cdots, n$.

$G$ is *n-chromatic* (or the *chromatic number* of $G$ is $n$), if $G$ is $n$-colorable but not $n'$-colorable for any $n'<n$.

The definition of $n$-colorable graphs in this paper is distinct from the one in [3],*) but the definitions of $n$-chromatic graphs in this paper and in [3] are equivalent.

In this paper a surface means a differentiable or combinatorial closed 2-manifold and an imbedding of $G$ into a surface $M$ means differentiable or piece-wise linear one, regarding $G$ as a complex. For definition of differentiable map of complex, see, for example, [2].

$G$ can be imbedded in some orientable surface having enough large number of genus. $G$ is *m-imbeddable* if it can be imbedded in an orientable surface having genus $m$, and is *minimal m-imbeddable* if it can be $m$-imbeddable but not $(m-1)$-imbeddable. It is said that the *genus* of $G$ is $m$ if $G$ is minimal $m$-imbeddable.

§ 2. **An imbedding theorem.** To express an imbedding of $G$ into a surface $M$ or the imbedded subspace of $M$, we use the notation $G(M)$. An imbedding $G(M)$ is said to be *simplest* if $\chi(M)\geqq\chi(N)$ for any imbedding $G(N)$, where $\chi(M)$ is the Euler characteristic of $M$. If any connected component of $M-G(M)$ is open 2-cell, $G(M)$ is said to be *2-cell imbedding*.

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*) See the references on the last page (p. 1024) of the part II of this paper.