222. A Note on Function-theoretic Null-sets of Class $N_{\mathfrak{S}\mathfrak{D}}$

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1. Let Ω be a plane region and let E be its complementary set with respect to the Riemann sphere. Following Ahlfors and Beurling [1], we shall say that E is of class $N_{\mathfrak{B}}$ (resp. $N_{\mathfrak{D}}$, $N_{\mathfrak{S}\mathfrak{B}}$, and $N_{\mathfrak{S}\mathfrak{D}}$), if Ω carries no non-constant analytic function in it which is bounded (resp. with finite Dirichlet integral, univalent and bounded, and univalent and with finite Dirichlet integral). They showed that $N_{\mathfrak{B}} \subsetneqq N_{\mathfrak{D}} \subsetneqq N_{\mathfrak{S}\mathfrak{B}} = N_{\mathfrak{S}\mathfrak{D}}$.

It is known that the union of a finite number of sets of class $N_{\mathfrak{B}}$ (resp. $N_{\mathfrak{D}}$) belongs to the same class. This was proved by Kametani [4] for mutually disjoint sets of class $N_{\mathfrak{B}}$ and later by Kuroda [5] for the same class without the restriction of disjointness. This is also true for the union of a countable number of these sets, so long as it is compact (see Noshiro [6], footnote, p. 11).

The class $N_{\mathfrak{S}\mathfrak{D}}$ does not have this property. We shall verify this by constructing a counterexample.

2. A boundary component α of a plane region is called *weak* [8], if its image under every conformal mapping is always a point. Among several properties of weak boundary components obtained by Grötzsch [2], Sario [7, 8], Jurchescu [3], and others, the following will be needed in the next section;

i) (Jurchescu [3]) The weakness is a boundary property. That is, if there exists a conformal mapping, denoted by $f(z)$, of a neighborhood of a weak boundary component α onto a neighborhood of $f(\alpha)$ of a region, then $f(\alpha)$ is weak with respect to the region.

ii) (Sario [7] and Jurchescu [3]) A compact set E is of class $N_{\mathfrak{S}\mathfrak{D}}$, if and only if each boundary component of E^c is weak.

iii) (Grötzsch [2]) α is weak, if there exists a sequence of mutually disjoint doubly connected regions $\{R_n\}$ such that R_{n+1} separates α from R_n and that the series $\sum \operatorname{mod} R_n$ diverges. Here $\operatorname{mod} R_n$ is the logarithm of the ratio of the outer and the inner radius of a conformally equivalent annulus of R_n.

By the properties i) and ii) we get immediately

Lemma. *The union of a finite number of mutually disjoint sets of class $N_{\mathfrak{S}\mathfrak{D}}$ is in the same class.*