246. Some Characterizations of Certain von Neumann Algebras

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In this note, we shall give some characterizations of finiteness and proper infiniteness of von Neumann algebras with respect to the density property of some subsets of von Neumann algebras.

1. Let $\mathfrak{H}$ be a Hilbert space and $\mathcal{L}(\mathfrak{H})$ the algebra of all (bounded linear) operators acting on $\mathfrak{H}$. By a *von Neumann algebra* $\mathcal{A}$ acting on $\mathfrak{H}$ we mean a weakly closed self-adjoint subalgebra (containing the identity I) of $\mathcal{L}(\mathfrak{H})$. A von Neumann algebra $\mathcal{A}$ is called *finite* if, for any $A \in \mathcal{A}$, $A^*A=I$ implies $AA^*=I$. This definition is equivalent to that, for any nonzero positive operator $A \in \mathcal{A}$, there exists a normal finite trace φ on $\mathcal{A}$ such that $\varphi(A)\neq 0$.

B. Russo and H. A. Dye [9] showed that, in any von Neumann algebra $\mathcal{A}$, the convex hull of the set of unitary operators in $\mathcal{A}$ is weakly dense in the unit sphere of $\mathcal{A}$. Using the technique of Russo and Dye, we have the following:

Lemma 1. *In any von Neumann algebra $\mathcal{A}$, the set of all unitary operators in $\mathcal{A}$ is strongly dense in the set of all isometric operators in $\mathcal{A}$.*

Proof. Let V be an isometric operator in $\mathcal{A}$ such that $VV^*=P\neq I$ and $\mathfrak{M}$ be the range of $I-P$. Then $\{V^n\mathfrak{M}\,;\, n\geqq 0\}$ are mutually orthogonal. Let

$$\mathfrak{N}=\left(\bigoplus_{n=0}^{\infty} V^n\mathfrak{M}\right)^{\perp}.$$

Then $\mathfrak{N}$ is invariant under V and V^*. If Q is the projection onto $\mathfrak{N}$, then Q is dominated by P and so that

$$VQV^*=VV^*Q=Q,$$

or the restriction of V to $\mathfrak{N}$ is unitary.

Given $\varepsilon>0$ and vectors $x_1, \cdots, x_n \in \mathfrak{H}$, choose m such that

$$\|\sum_{k>m} Q_k x_i\|<\varepsilon/2,$$

for all i, where Q_k is the projection onto $V^k\mathfrak{M}$. Let

$$U=\begin{cases} V \text{ on } \mathfrak{N}+\mathfrak{M}+V\mathfrak{M}+\cdots+V^m\mathfrak{M} \\ (V^{m+1})^* \text{ on } V^{m+1}\mathfrak{M} \\ I \text{ on } \bigoplus_{k>m+1} V^k\mathfrak{M}. \end{cases}$$

Then U is a unitary operator in $\mathcal{A}$ such that