# 182. A Sharp Form of Gårding's Inequality for a Class of Pseudo-Differential Operators

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**Introduction.**

The present paper is concerned with an algebra of a class of pseudo-differential operators and a sharp form of Gårding's inequality (see [3], [4], [5], and [7]), which are important for the study of the Cauchy problem for pseudo-differential equations of parabolic type. Let $\lambda(\xi)$ be a basic weight function which means that $\lambda(\xi)$ is a real valued $C^\infty$-function satisfying $\lambda(\xi)\geqq 1$ and $|\partial_\xi^\alpha\lambda(\xi)|\leqq C_\alpha\lambda(\xi)^{m-|\alpha|}$ (see [5]).

Then we say $p(x,\xi)\in S^m_{0,\lambda}$ when $p(x,\xi)\lambda(\xi)^{-m}\in\mathscr{B}(R^{2n}_{(x,\xi)})$.[1]

For $p(x,\xi)\in S^m_{0,\lambda}$, the pseudo-differential operator $P$ is defined by

$$Pu(x)=p(X,D_x)u(x)=\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)đ\xi$$

where $đ\xi=(2\pi)^{-n}d\xi$ and $\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx$.

From Theorem 1.1, we have

(1) *Let* $p(x,\xi)\in S^m_{0,\lambda}$. *Then there exists* $p^*(x,\xi)\in S^m_{0,\lambda}$ *such that*

$$(p(X,D_x)u,v)=(u,p^*(X,D_x)v)\qquad \textit{for any } u,v\in\mathcal{S}.$$

(2) *Let* $p_j(x,\xi)\in S^{m_j}_{0,\lambda}$ $(j=1,2)$. *Then there exists* $p_{1,2}(x,\xi)\in S^{m_1+m_2}_{0,\lambda}$ *such that*

$$p_{1,2}(X,D_x)u=p_1(X,D_x)\cdot p_2(X,D_x)u\qquad \textit{for any } u\in\mathcal{S}.$$

These properties mean that the operator class corresponding to $S^\infty_{0,\lambda}=\bigcup_{-\infty<m<\infty}S^m_{0,\lambda}$ forms an algebra.

Recently Calderón and Vaillancourt [1] proved the $L^2$-boundedness $\|Pu\|_{L^2(R^n)}\leqq C\|u\|_{L^2(R^n)}$ for $p(x,\xi)\in S^0_{0,\lambda}=\mathscr{B}(R^{2n}_{(x,\xi)})$. Using this estimate essentially, we obtain the inequality

$$\|Pu\|_s\leqq C\|u\|_{s+m}\qquad \text{for } p(x,\xi)\in S^m_{0,\lambda}$$

where $\|u\|_s^2=\|u\|_{s,\lambda}^2=\int\lambda(\xi)^{2s}|\hat{u}(\xi)|^2đ\xi$ (see Corollary 1.2).

From this inequality and the Friedrichs' approximation we can derive a sharp form of Gårding's inequality

$$\mathcal{R}e(p(X,D_x)u,u)\geqq -C\|u\|^2_{1/2(m-1)}$$

when $p(x,\xi)\geqq 0$ and $\partial_\xi^\alpha p(x,\xi)\in S^{m-|\alpha|}_{0,\lambda}$ for $|\alpha|\leqq 2$ (Theorem 2.3 (2)).

The proofs of our results are based on the method in Kumano-go [4].

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1) $\mathscr{B}(R^N)=\{u\in C^\infty(R^N);|\partial_x^\alpha u(x)|\leqq C_\alpha \text{ for any } \alpha\}$.