207. An Application of A. Robinson's Proof of the Completeness Theorem

By Yoshihiro NAKANO
Yamaguchi University

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In [1], A. Robinson gave a proof of the Gödel's Completeness Theorem. Robinson's argument seems to be applicable in other branches. In this note, we shall give an application in universal algebra. There is a theorem, called Grätzer's theorem, which argues a kind of finiteness property with respect to the existence of homomorphisms (cf. [2], p. 138). The usual proof of this theorem which uses the condition for the inverse limit to be non-void is rather complicated. We shall give a simplified direct proof of the extended version of this theorem, which is a slight modification of Robinson's argument.

In the following, we shall use the usual set-theoretical notation. Thus, a mapping $\varphi: A \to B$ is a subset of $A \times B$ satisfying certain conditions. For a mapping φ, $\mathrm{Dom}(\varphi)$ denotes the domain of φ and $\mathrm{Ra}(\varphi)$ the range of φ. If C is a set, then $\varphi | C$ denotes the restriction of φ to $C \cap \mathrm{Dom}(\varphi)$.

A structure $\mathfrak{A} = \langle A, R_j^{\mathfrak{A}} \rangle_{j \in J}$ is a set A together with an indexed set $\{R_j^{\mathfrak{A}}\}_{j \in J}$ of finitary operations and relations on A. Then A is the universe of $\mathfrak{A}$. In the following, we shall consider only the relational systems, that is the structure with no operation. This restriction entails no loss of generality, because the structures can be considered as the relational systems by replacing the operations by their graphs. Then, our version of the Grätzer's theorem is the following:

Let $\mathfrak{A} = \langle A, R_j^{\mathfrak{A}} \rangle_{j \in J}$ be a finite relational system. A relational system $\mathfrak{B} = \langle B, R_j^{\mathfrak{B}} \rangle_{j \in J}$ has a homomorphism into $\mathfrak{A}$ if and only if every finite subsystem of $\mathfrak{B}$ has a homomorphism into $\mathfrak{A}$.

The "only if" part is trivial. To prove "if" part, assume that every finite subsystem of $\mathfrak{B}$ has a homomorphism into $\mathfrak{A}$, and let Σ be the set of all homomorphisms of finite subsystems of $\mathfrak{B}$ into $\mathfrak{A}$. Now, let Ω be the set of mappings φ from subsets of B into A satisfying the following condition:

$$(*)\begin{cases}\text{For every finite subset } C \text{ of } B\text{, there is a}\\ \text{homomorphism } \psi \text{ in } \Sigma \text{ such that the domain}\\ \text{of } \psi \text{ is } C \text{ and } \varphi | C = \psi | \mathrm{Dom}(\varphi).\end{cases}$$

Since every finite subsystem of $\mathfrak{B}$ has a homomorphism into $\mathfrak{A}$, the