123. On Power Cancellative Archimedian Semigroups

By Naoki KIMURA and Yen-Shung TSAI
University of Arkansas

(Comm. by Kenjiro SHODA, M. J. A., Sept. 12, 1972)

The following problem was raised by Tamura*)

Problem. Is a power cancellative archimedian semigroup necessarily cancellative?

The purpose of this short note is to give an affirmative answer to this problem. Throughout the paper a semigroup means a commutative semigroup, and a subsemigroup and an ideal are assumed non-empty. A semigroup is archimedian if for any two elements x, y, some power of x is divisible by y. Any terminology not defined here should be referred to [1].

Lemma 1. *Every ideal of an archimedian semigroup is an absorbing subset, in the sense that, every element has some power contained in the ideal.*

Proof. Let A be an archimedian semigroup and B an ideal of A. Let $b \in B$. Since A is archimedian, for every element $a \in A$ there exist a positive integer n and an element $c \in A$ such that $a^n = bc$. But $bc \in B$, because B is an ideal. Hence it follows that $a^n \in B$.

Remark 1. The condition in the lemma is actually the necessary and sufficient condition for a semigroup to be archimedian.

A semigroup is called power cancellative, if $x^n = y^n$ for some positive integer n always implies $x = y$.

Theorem 1. *Every power cancellative archimedian semigroup is cancellative.*

Proof. Let A be a power cancellative archimedian semigroup. Assume that $ac = bc$, where $a, b, c \in A$. Consider the subset C of A defined by $C = \{x \in A \| ax = bx\}$. Then C is not empty, because $c \in C$, and it is easily seen that C is an ideal of A. Hence it is absorbing by Lemma 1. Therefore there exist positive integers m and n such that a^m, $b^n \in C$. From the definition of C, it follows immediately that $a^k x = b^k x$ for every element $x \in C$ and for every positive integer k.

Therefore we have

$$a^{m+n} = a^m b^n = b^{m+n}.$$

Since A is power cancellative, it follows that $a = b$.

This completes the proof of the theorem.

Remark 2. A cancellative archimedian semigroup may not be

*) In his letter to one of the authors.