3. *On the Representations of Semi-Simple Lie Groups*

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(Comm. by K. KUNUGI, M.J.A., Jan. 12, 1954)

The purpose of this note is to give some remarks on the representations of semi-simple Lie groups. In this note we give only the Definitions and Theorems, since we shall give discussions elsewhere in detail.

Let G be a connected Lie group, and $C_c^\infty(G)$ be the algebra composed of indefinitely differentiable complex-valued functions with compact supports.

Let $U(G)$ be the subalgebra of $D(G)$ composed of all elements whose supports reduce to the identity, then $U(G)$ is isomorphic to the universal enveloping algebra B[1] corresponding to G.

Let $D_z(G)$ be the center of $D(G)$ and $\varepsilon_s(s \in G)$ be the point measure with mass 1 at s.

Then we can easily show that $\alpha(\in D(G))$ belongs to $D_z(G)$ if and only if $\varepsilon_s \alpha \varepsilon_{s^{-1}} = \alpha$ for all $s \in G$. Let $U_z(G)$ be the center of $U(G)$, then $U_z(G) \subset D_z(G)$.

Let $\{\Pi(x), \mathfrak{H}\}$ be a strongly continuous representation of G on a Banach space $\mathfrak{H}$ and $\{\Pi(f), \mathfrak{H}\}$ be the corresponding representation of $C_c^\infty(G)$. Let $\boldsymbol{B}$ be the operator algebra composed of all bounded operators on $\mathfrak{H}$. We shall state

Definition 1. A representation $\{\Pi(x), \mathfrak{B}\}$ is n-fold irreducible, if there exists an element $\Pi(f)$ such that

$$\|\Pi(f)x_i - Bx_i\| < \varepsilon \qquad (i = 1, 2, \dots, n)$$

for arbitrary at most n elements $x_1, \dots, x_n$, $B \in \boldsymbol{B}$ and $\varepsilon > 0$.

Proposition. If $\{\Pi(x), \mathfrak{H}\}$ is 2-fold irreducible, it is quasi-simple.[2]

In the following, we shall suppose that G is a connected semi-simple Lie group with a decomposition $G = K \cdot S (K \cap S = (e))$ where K is a maximal compact subgroup and S is a quasi-nilpotent subgroup[3] in the sense of Harish-Chandra.[3] Since the above condition i.e. $G = K \cdot S$, seems to be indispensable at certain essential points in our note, we have decided for the sake of uniformity to assume it throughout. Let P be the set of all equivalence classes of irreducible representations of K and $\chi_d(k)$ be the character of $d(\in P)$.

We shall denote the equivalence class of irreducible representation of $U(K)$ which corresponds to $d(\in P)$ by the same notation d.