115. On a Generalization of Groups

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A group can be characterized as a multiplicative system with an operator θ satisfying the following three conditions:

$$\text{I} \qquad (ab)c=a(bc),$$

$$\text{II} \qquad (a^\theta a)b=b,$$

$$\text{III}' \qquad a^\theta a=b^\theta b.$$

Now let us consider about a multiplicative system G with an operator θ satisfying I, II and

$$\text{III} \qquad (ab)^\theta=b^\theta a^\theta.$$

We shall call this a G-system. Then a group is a G-system satisfying $a=a^{\theta\theta}$ for any element a. In this note we shall prove that a G-system is a product of a group and a subsystem consisting of all idempotents.

We shall firstly prove some properties about the operator θ.

1. $a^{\theta\theta\theta}=a^\theta$.

Proof. From II we obtain $a^\theta ab=b$. Multiplying the both sides by $a^{\theta\theta}$ from the left, we have $ab=a^{\theta\theta}b$ by II and $b^\theta a^{\theta\theta\theta}=b^\theta a^\theta$ by III. Multiplying the both sides by $b^{\theta\theta}$ from the left, we have $a^{\theta\theta\theta}=a^\theta$.

2. $ex=x$ and $e^\theta=e$ for $e=a^{\theta\theta}a^\theta$.

Proof. $ex=(a^\theta)^\theta a^\theta x=x$, $e^\theta=(a^{\theta\theta}a^\theta)^\theta=a^{\theta\theta}a^{\theta\theta\theta}=a^{\theta\theta}a^\theta=e$.

3. $a^{\theta\theta}a^\theta=b^{\theta\theta}b^\theta$.

Proof. $b^\theta=(eb)^\theta=b^\theta e^\theta=b^\theta e$, hence $b^{\theta\theta}b^\theta=b^{\theta\theta}b^\theta e=e$.

4. $a^\theta a^{\theta\theta}=a^{\theta\theta}a^\theta$.

Proof. Putting $b=a^\theta$ in 3 we obtain $a^{\theta\theta}a^\theta=a^{\theta\theta\theta}a^{\theta\theta}=a^\theta a^{\theta\theta}$.

5. $xe=x^{\theta\theta}$.

Proof. If we put $y=xe$, then $x^\theta xe=x^\theta y$ and $e=x^\theta y$. Therefore $y=x^{\theta\theta}x^\theta y=x^{\theta\theta}e=x^{\theta\theta}x^\theta x^{\theta\theta}=x^{\theta\theta}$.

6. $e=aa^\theta$.

Proof. $ae=a^{\theta\theta}$ by 5. Multiplying the both sides by a^θ from the right, we have $aea^\theta=a^{\theta\theta}a^\theta=e$. On the other hand, $aea^\theta=a(ea^\theta)=aa^\theta$.

Since θ is an anti-endomorphism of G and the condition III′ holds in G^θ by 3, G^θ is a group. Let $\{C(a)\}$ be the set of classes $C(a)$ of G induced by the anti-endomorphisms θ, where $C(a)$ is the class involving a. Then the set forms a group anti-isomorphic to G^θ.

Theorem 1. *$C(e)$ is a set of all idempotents in G.*

Proof. II implies $a^\theta a^2=a$, therefore $a^\theta a=a$, $a^\theta=(a^\theta a)^\theta=a^\theta a^{\theta\theta}=e$