127. On the B-covers in Lattices

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Let L be a lattice. For any two elements a and b of L we shall define the following three kinds of sets:

(1) $$J(a,b)=\{x\,|\,x=(a\frown x)\smile(b\frown x)\}$$

(2) $$C(J(a,b))=\{x\,|\,x=(a\smile x)\frown(b\smile x)\}$$

(3) $$B(a,b)=J(a,b)\frown C(J(a,b)).$$

$B(a,b)$ is called the B-cover of a and b. If $c\in B(a,b)$, we shall write acb simply.

In case L is a normed lattice, a point c is defined to be between two points a and b if $d(a,c)+d(c,b)=d(a,b)$, where $d(x,y)=|x\smile y|-|x\frown y|$. Several lattice characterizations of this metric betweeness have been obtained by V. Glivenko [1], L. M. Blumenthal and D. O. Ellis [2] and the author [3]; namely c lies between a and b in the metric sense if and only if one of the following conditions is satisfied in the associated normed lattice L.

(G) $$(a\frown c)\smile(b\frown c)=c=(a\smile c)\frown(b\smile c)$$

(G*) $$(a\frown c)\smile(b\frown c)=c=c\smile(a\frown b)$$

(G**) $$(a\smile c)\frown(b\smile c)=c=c\frown(a\smile b)$$

(M) $$(a\smile(b\frown c))\frown(b\smile c)=c.$$

Thus our definition of "acb" in an arbitrary lattice is a generalization of metric betweeness in a normed lattice. The notion of B-cover for a normed lattice is due to L. M. Kelley [4].

In Theorem 1 we shall assert that $(a]\smile(b]=J(a,b)\subset(a\smile b]$, $[a)\frown[b)=C(J(a,b))\subset[a\frown b)$. In Theorem 2 we shall deal with the relations between the two B-covers $B(a,b)$ and $B(b,c)$.

In Theorem 3 we shall consider the necessary and sufficient condition (A) in order that L be a distributive lattice.

In Theorems 4 and 5, we shall give the structures of $B(a,b)$ by imposing algebraic restrictions on them. Theorem 4 gives a generalization of the important result obtained by L. M. Kelley.

Now let $x\in J(a,b)$, then we have $x\geqq x\frown(a\smile b)\geqq(a\frown x)\smile(b\frown x)=x$, hence we obtain $x\frown(a\smile b)=(a\frown x)\smile(b\frown x)$, that is $(a,x,b)D$. From $x\frown(a\smile b)=x$, we get $x\leqq a\smile b$. We have clearly $a\smile b\in J(a,b)$, and $x\in J(a,b)$ if $x\leqq a$ or $x\leqq b$. On the other hand any element x of $J(a,b)$ is represented by $x=(a\frown x)\smile(b\frown x)$, where $a\frown x\in(a]$, $b\frown x\in(b]$. If we take any two elements x,y from $J(a,b)$, then $x\smile y$ belongs to $J(a,b)$. Indeed we have $x\smile y=(a\smile b)\frown(x\smile y)\geqq(a\frown(x\smile y))\smile(b\frown(x\smile y))$