26. *The Initial Value Problem for Linear Partial Differential Equations with Variable Coefficients. II*

By Taira SHIROTA

(Comm. by K. KUNUGI, M.J.A., Feb. 12, 1957)

In the present note we give another simple proof of Theorem 2 in Paper I using the duality of Hilbert space defined over $R^1_t \times R^l_x$. This idea owes to Prof. M. Nagumo who develops our theorem in an abstract form,[1] but it seems good to me to refine his postulate.

Let $A\left(t, x, \dfrac{\partial}{\partial x}\right)$ be an (m, m)-smooth system defined on $R^1_t \times R^l_x$ and let $B\left(t, x, \dfrac{\partial}{\partial x}\right)$ be an other (m, m)-smooth system which is uniformly strongly elliptic with sufficiently large $s=\{s(i) \mid i=1,2,\cdots,m\}$ for any $t \in R^1_t$.

Lemma 1. *For sufficiently large integers s' and s'' let $\mathfrak{D}^{(s')}_t(H^{(s'')}_x)$ be a space of all s'-time differentiable vector valued functions on R^1_t into $H^{(s'')}_x$*[2] *with compact carriers. Then for some integer $k(s'')$ the differential operator $B\left(t, x, \dfrac{\partial}{\partial x}\right)+B\left(t, x, \dfrac{\partial}{\partial x}\right)^*+k(s'')$ has the inverse from $\mathfrak{D}_{t,x}$ into $\mathfrak{D}^{(s')}_t(H^{(s'')}_x)$.*

From Lemma 1 and Sobolev's lemma $(B+B^*+k(s''))^{-1}(\mathfrak{D}_{t,x})$ is contained in the space of functions defined over $R^1_t \times R^l_x$ with derivatives of orders s for some $s<s' \wedge s''$.

Lemma 2. *Let A be semi-bounded by the norm defined by B in the strong sense, i.e.,*

$$((A_t u, u))_{B_t} \leqq \gamma((u, u))_{B_t} \qquad \textit{for } u \in \mathfrak{D}_x$$

for some positive constant γ. Then for any $u \in \mathfrak{D}_{t,x}$ the following inequalities hold:

$$(1) \qquad \int_{-\infty}^{\infty}\left(\left(\left(\frac{\partial}{\partial t}-\bar{A}_t\right)u_t, u_t\right)\right)_{B_t} dt \geqq \beta \int_{-\infty}^{\infty}((u_t, u_t))_{B_t} dt,$$

$$(2) \qquad \| e^{\delta t} u_t \|_{B_t} \leqq \| e^{\delta t_0} u_{t_0} \|_{B_{t_0}} + \left\{\int_{t_0}^{t} \left\| e^{\delta \tau}\left(\frac{\partial}{\partial \tau}-\bar{A}_\tau\right)u_\tau \right\|_{B_\tau} d\tau\right\} \quad \textit{for } t>t_0$$

where $\bar{A}=A-\alpha$, α, β, δ are some positive reals.

The inequality (1) implies the following:

$$(3) \qquad \int_{-\infty}^{\infty}\left\|\left(\frac{\partial}{\partial t}-\bar{A}\right)u_t\right\|^2_{B_t} dt \geqq \beta^2 \int_{-\infty}^{\infty} \| u_t \|^2_{B_t} dt$$

1) M. Nagumo: On linear hyperbolic system of partial differential equations in the whole space, Proc. Japan Acad., **32** (1956).

2) $H^{(s)}_x$ is H_s in Paper I.