143. *An Extension Theorem*

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In an earlier paper [3], we obtained an extension theorem ([3, Theorem 2.2]; [4, § 79 Theorem 2]) about linear functionals on modulared linear spaces. From the proof of this theorem we conclude immediately:

Extension Theorem. *Let R be a linear space and m a functional on R subject to*

1) $\qquad 0 \leqq m(x) \leqq +\infty \quad$ *for every* $x \in R$,

2) $\qquad m(\lambda x+\mu y) \leqq \lambda m(x)+\mu m(y) \quad$ *for* $\lambda+\mu=1;\ \lambda, \mu \geqq 0$.

For a linear manifold A of R, a linear functional φ on A and a real number γ, if

$$\varphi(x) \leqq \gamma+m(x) \quad \text{for every } x \in A,$$

then we can find a linear functional ψ on R such that

$$\psi(x)=\varphi(x) \quad \text{for every } x \in A,$$
$$\psi(x) \leqq \gamma+m(x) \quad \text{for every } x \in R.$$

As an application of this extension theorem, we will prove here Ascoli's theorem [1, 2]: every closed convex set in a Banach space also is weakly closed. Using the terminologies in the book [4], we state this theorem in more general form:

Theorem. *Let R be a convex linear topological space, and A a closed convex set. For any $a \,\overline{\in}\, A$, we can find a continuous linear functional φ on R such that*

$$\varphi(a) > \sup_{x \in A} \varphi(x).$$

Proof. We can suppose $0 \in A$ without loss of generality. Since A is closed, for any $a \,\overline{\in}\, A$ we can find a convex pseudo-norm for ichwh

$$\inf_{x \in A} \| x-a \| > 0.$$

For such a convex pseudo-norm, putting

$$m(x)=\inf_{y \in A} \| x-y \|,$$

we see easily that $0 \leqq m(x) < +\infty$; $m(x)=0$ for $x \in A$, and

$$m(\lambda x+\mu y) \leqq \lambda m(x)+\mu m(y)$$

for $\lambda+\mu=1;\ \lambda, \mu \geqq 0$, because A is convex. Furthermore, putting

$$\varphi_0(\xi a)=\xi m(a),$$
$$\gamma=\sup_{0 \leqq \xi \leqq 1} \{\xi m(a)-m(\xi a)\},$$

we obtain a linear functional φ_0 on the linear manifold

$$\{\xi a;\ -\infty < \xi < +\infty\},$$