142. On Yamamuro's Theorem Concerned with Linear Modulars

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Some years ago S. Yamamuro [1] proved the following theorem which is the first success in the special case of the so-called norm problem in the theory of modulared semi-ordered linear spaces inaugurated by H. Nakano [2].

Theorem. Let R be a semi-regular modulared semi-ordered linear space with a modular m. In order that the first norm $\|a\|$ and the second norm $|||a|||$ defined by m coincide for every $a \in R$, it is necessary and sufficient that the modular m is either linear or singular.

Since then, I. Amemiya obtained the following convenient formula for the first norm:

$$(1) \qquad \|a\| = \inf_{\xi>0} \frac{1+m(\xi a)}{\xi},$$

and, by systematic uses of this formula, he established a characterization of the modulars of L_p type ($p>1$) in [3] which, together with above Yamamuro's theorem, constitutes a sufficiently general solution of the norm problem.

The purpose of this short note is to show that Amemiya's method is also applicable for the above theorem and we can obtain its extremely simple proof without use of conjugate modulars.

Since the sufficiency is well known (cf. [2, Theorems 41.3 and 41.4]), we shall prove the necessity dividing in three steps in follows. In the sequel, we suppose that R is a semi-regular modulared semi-ordered linear space in which the two norms defined by its modular always coincide.

1. *For any $a \in R$ such that $0<m(a)<+\infty$ there exists a real number $0<\alpha<+\infty$ for which we have $m(\alpha a)=1$.*

Since the above statement is evidently true by modular conditions if there exists a real number $0<\beta<+\infty$ such that $1 \leqq m(\beta a)<+\infty$, we suppose that

$$m(\xi a)<1 \text{ for } 0 \leqq \xi \leqq \gamma, \text{ and } m(\xi a)=+\infty \text{ for } \xi>\gamma$$

for some real number $0<\gamma<+\infty$. Then, by the formula (1), there exists a real number $0<\xi_0<+\infty$ for which we have

$$0<\xi_0 \leqq \gamma, \quad \|a\| = \frac{1+m(\xi_0 a)}{\xi_0}.$$

On the other hand we have