9. On Eigenfunction Expansions of Self-adjoint Ordinary Differential Operators. II

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§3. We introduce the *characteristic matrix* of H by

$$\begin{aligned} M_{11}&=f_a(l)\cdot f_b(l)[f_a(l)-f_b(l)]^{-1}\\ M_{12}&=M_{21}=(1/2)[f_a(l)+f_b(l)][f_a(l)-f_b(l)]^{-1}\\ M_{22}&=[f_a(l)-f_b(l)]^{-1}\end{aligned} \tag{10}$$

where $f_a(l)$, $f_b(l)$ are the characteristic functions of H. By (1), M_{jk} $(j, k=1, 2)$ are regular on the upper and the lower half complex planes $(\Im l \neq 0)$ and

$$M_{jk}(\bar{l})=\overline{M_{jk}(l)} \quad (j, k=1, 2). \tag{11}$$

For every real number λ, the limits

$$\rho_{jk}(\lambda)=\lim_{\substack{\delta\to+0\\ \delta'\to+0}}\lim_{\varepsilon\to+0}\pi^{-1}\int_{\delta}^{\lambda+\delta'}\Im M_{jk}(\lambda+i\varepsilon)\,d\lambda \tag{12}$$

exist.[1]

As a function of λ, the matrix function $p(\lambda)=(\rho_{jk}(\lambda))$ is continuous on the right and monotone non-decreasing in the sense that, for $\mu<\lambda$, the symmetric matrix $p(\lambda)-p(\mu)$ is positive semi-definite.[2] Hence by the well-known procedure we can construct the matrix set function $p(B)=(\rho_{jk}(B))$ of bounded Borel sets B on the real line corresponding to $p(\lambda)$. $p(B)$ is positive semi-definite, and completely additive on every bounded Borel set. For every $\nu>0$, the residual terms

$$R_{jk}^{(\nu)}(l)=M_{jk}(l)-\int_{-\nu}^{\nu}(\lambda-l)^{-1}\,d\rho_{jk}(\lambda) \tag{13}$$

are regular in the l-plane except for real l such that $l\leqq-\nu$ or $l\geqq\nu$.[2] For the transformation (2) of the system of fundamental solutions, $\rho_{jk}(\lambda)$ are transformed as follows

$$\rho_{jk}(\lambda)=\int_0^{\lambda}\sum_{m,n}\beta_{mj}(\lambda)\,\beta_{nk}(\lambda)\,d\tilde{\rho}_{mn}(\lambda).^{3)} \tag{14}$$

By (11), (12) and the regularity of $M_{jk}(l)$ for $\Im l\neq 0$, we have for $\lambda'>\lambda$

$$\rho_{jk}(\lambda')-\rho_{jk}(\lambda)=-\lim_{\substack{\mu\to\lambda+0\\ \mu'\to\lambda'+0}}\lim_{\varepsilon\to+0}(2\pi i)^{-1}\int_{C(\mu',\mu,\alpha,\varepsilon)}M_{jk}(l)\,dl \tag{15}$$

where $C(\mu', \mu, \alpha, \varepsilon)$ means the contour consisting of two oriented polygonal lines whose vertices, in order, are $\mu'+i\varepsilon$, $\mu'+i\alpha$, $\mu+i\alpha$, $\mu+i\varepsilon$

1) Cf. Kodaira [3], Theorem 1.3.
2) Cf. Kodaira [3], Theorem 1.3.
3) Cf. Kodaira [3], p. 932.