26. *Note on Idempotent Semigroups. III*

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§ 1. This note is an abstract of the paper (Naoki Kimura [1]), in which the author proved the structure theorems of some special idempotent semigroups. Terminologies in the previous papers (Kimura [2] and Miyuki Yamada and Kimura [3]) will be used without definitions.

§ 2. An idempotent semigroup is called

(1) *right* (*left*) *semi-regular* if $axy=axyayxy$ ($xya=xyxaxya$),

(2) *right* (*left*) *semi-normal* if $axy=axyay$ ($xya=xaxya$),

(3) *right* (*left*) *quasi-normal* if $axy=axay$ ($xya=xaya$),

for all a, x, y.

Then the following implications are easy to prove:

$$\begin{tikzcd}
\text{r. regular} \arrow[r] & \text{r.q.-normal} \arrow[r] \arrow[dr] & \text{r.s.-normal} \arrow[r] & \text{r.s.-regular} \\
\text{normal} \arrow[ur] \arrow[dr] & & \text{regular} \arrow[ur] \arrow[dr] & \\
\text{l. regular} \arrow[r] & \text{l.q.-normal} \arrow[r] \arrow[ur] & \text{l.s.-normal} \arrow[r] & \text{l.s.-regular.}
\end{tikzcd}$$

Further, we have the following lemmas:

LEMMA 1. *An idempotent semigroup is regular if and only if it is both left and right semi-regular.*

LEMMA 2. *An idempotent semiproup is normal if and only if it is both left and right quasi-normal* (*semi-normal*).

§ 3. Let $\mathfrak{P}(\mathfrak{Q})$ be the equivalence on an idempotent semigroup S defined by

$$x\mathfrak{P}y \text{ if and only if } xy=y \text{ and } yx=x,$$
$$x\mathfrak{Q}y \text{ if and only if } xy=x \text{ and } yx=y.$$

Then we have the following representation theorems.

THEOREM 1. *$\mathfrak{P}(\mathfrak{Q})$ is a congruence on an idempotent semigroup S if and only if S is left* (*right*) *semi-regular. Further, in this case the quotient semigroup $S/\mathfrak{P}(S/\mathfrak{Q})$ is left* (*right*) *regular.*

From this theorem and Lemma 1, we have

COROLLARY. *Both $\mathfrak{P}$ and $\mathfrak{Q}$ are congruences on an idempotent semigroup S if and only if S is regular. Further, in this case S is isomorphic to the spined product of $S/\mathfrak{P}$ and $S/\mathfrak{Q}$ with respect to its structure semilattice.*

REMARK. This corollary is essentially the same as Theorem 2 in [2], and the above method gives an alternative proof for it.