81. Relations between Solutions of Parabolic and Elliptic Differential Equations

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In this note we shall show that under some conditions the solution $u(x,t)$ of

$$\sum_{i=1}^{m}\frac{\partial^2 u}{\partial x_i^2}-\frac{\partial u}{\partial t}=f(x,t,u)$$

converges to a solution $v(x)$ of

$$\sum_{i=1}^{m}\frac{\partial^2 v}{\partial x_i^2}=\bar{f}(x,v)$$

as $t\to\infty$.

Let G be a domain which is regular for Laplace's equation[1] in the m-dimensional Euclidean space, and let Γ be the boundary of G. Set $D=G\times(0,\infty)$ and $B=\Gamma\times[0,\infty)$. We remark that D is regular for the heat equation[2] and therefore regular for the equation (E_1) below.[3]

Now, let $\mho$ and $\triangle$ be the generalized heat operator[4] and the generalized Laplacian operator respectively, i. e.

$$\mho u(x,t)=\lim_{r\downarrow 0}\frac{(n+2)^{\frac{m}{2}+1}}{m\pi^{\frac{m}{2}}r^2}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{u(\xi,\tau)-u(x,t)\}\sin^{m-1}\theta$$
$$\times\cos\theta\,(\log\operatorname{cosec}\theta)^{\frac{m}{2}}J\,d\varphi_1\cdots d\varphi_{m-1}d\theta$$

and

$$\triangle u(x)=\lim_{r\downarrow 0}\frac{2\cdot\Gamma\left(\frac{m}{2}+1\right)}{\pi^{\frac{m}{2}}r^2}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{u(\xi)-u(x)\}J\,d\varphi_1\cdots d\varphi_{m-1},$$

where in the first expression, $(\xi,\tau)=(\xi_1,\cdots,\xi_m,\tau)$ with

$$\xi_i=x_i+2r\sqrt{m}\sin\theta\sqrt{\log\operatorname{cosec}\theta}\,\eta_i\qquad(i=1,\cdots,m)$$

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1) This means that the 1st boundary value problem of Laplace's equation for G is always solvable for any continuous data on Γ.

2) "Regular for the heat equation" means that the 1st boundary value problem of the heat equation for D is always solvable for any continuous data on $G\smile B$. D is regular for the heat equation if and only if G is regular for Laplace's equation. For the proof, see "On the regularity of domains for parabolic equations", Proc. Japan Acad., **34**, 347-348 (1958).

3) It was proved in [1, p. 626] that a p-domain is regular for (E_1) if and only if it is regular for the heat equation.

4) See [1, p. 627], in which we used the symbol $\square$ instead of $\mho$.