1. *On the Singular Integrals.* V

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1. In the previous two papers [2, III, IV], we have studied the Hilbert transform from a point of view of the interpolation of operation and its applications. In [2, III] we have given a negative example as to the existence of this transformation, so we introduce a modified definition for a function of the more extensive class. In the book of N. I. Achiezer [1, p. 126] we find a modified definition, but this definition does not seem to be appropriate for the case $p>2$, because in the class L^p $(p>2)$ the Fourier transform does not necessarily exist. Here we introduce a new definition — a generalized Hilbert transform of order r:

$$\tilde{f}_r(x)=\frac{(x+i)^r}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(t+i)^r}\frac{dt}{x-t}, \tag{1.1}$$

where r is any positive real number.

In particular $\tilde{f}_0(x)$ means the ordinary one. Let $f(x)$ belong to L^p $(p\geqq 1)$ and $r=n$ $(n=1,2,\cdots)$. Then we have

$$\tilde{f}_n(x)=\tilde{f}_0(x)+C_{n-1}(x+i)^{n-1}+\cdots+C_0, \tag{1.2}$$

where

$$C_n=\int_{-\infty}^{\infty}\frac{f(t)}{(t+i)^{n+1}}dt,\quad (n=0,1,2,\cdots). \tag{1.3}$$

The present paper consists of two parts. In the first part we shall treat the integrability of (1.1) after [2, III]. In the second part we shall prove the reciprocal formula, and this plays an essential role in the study of the analytic function in a half-plane, as before [2, IV].

Chapter I. Integrability of the generalized Hilbert transform

2. Let $f(x)$ be a real or complex valued measurable function over $(-\infty,\infty)$. In order to make some variety we introduce the measure function as before

$$\mu(\alpha,x)=\int_0^x\frac{dt}{1+|t|^\alpha}\quad (0\leqq\alpha<1). \tag{2.1}$$

By L_μ^p $(p\geqq 1)$ we will denote the class of functions such that

$$\Big(\int_{-\infty}^{\infty}|f(x)|^p d\mu(\alpha,x)\Big)^{\frac{1}{p}}=\Big(\int_{-\infty}^{\infty}|f(x)|^p d\mu\Big)^{\frac{1}{p}}<\infty. \tag{2.2}$$

Then if we put