50. Between-topology on a Distributive Lattice

By Yatarō MATSUSHIMA
Gunma University, Maebashi

(Comm. by K. KUNUGI, M.J.A., May 7, 1959)

1. It is well known that the interval topology of a lattice L is defined by taking the closed intervals $[a)=\{x \mid x \geqq a\}$, $(a]=\{x \mid x \leqq a\}$ and $[a, b]=\{x \mid a \leqq x \leqq b\}$ as a sub-basis for closed sets. In [1–2] we have considered the concept of *B-covers* in lattices. For any two elements a and b of a lattice L, let

$B(a, b)=\{x \mid (a \smile x) \frown (b \smile x)=x=(a \frown x) \smile (b \frown x)\}$; then $B(a, b)$ is called the *B-cover* of a and b, and we write axb when $x \in B(a, b)$. Let $B^*(a,b)=\{x \mid abx\}$.

Now we shall define the *between-topology* on L as follows. By the *B-topology* (*B^*-topology*) of a lattice L, we mean that defined by taking the sets $B(a, b)$ ($B^*(a, b)$) as a sub-basis of closed sets.

In Theorem 1 we shall prove that the *B-topology* coincides with the interval topology in case L is a distributive lattice with O, I. It is shown in Theorem 2 that L_0 is a topological lattice in its *B^*-topology* when L_0 is a distributive lattice such that for any subset $B(a, b)$ of L_0, if $x, y \in B(a, b)$, then $a \frown x$ and $a \frown y$; $b \frown x$ and $b \frown y$ are comparable respectively.

E. S. Wolk [5] has defined that a subset X of a lattice L is *diverse* if and only if $x \in S$, $y \in S$, and $x \neq y$ imply that x and y are non-comparable. He showed that if L contains no infinite *diverse* set then L is a Hausdorff space in its interval topology.

Now we shall consider a distributive lattice L_0 with O, I satisfying the same assumption as in Theorem 2. Then in Theorem 3 we shall prove, by using the concept of the *B-covers* instead of that of *diverse* sets, that a certain type of L_0 is a Hausdorff space in its interval topology. This theorem is concerned with the Problem 23 of Birkhoff [3].

A *mob* is defined as a Hausdorff space with a continuous associative multiplication. In Theorem 4 we shall show that a distributive lattice L_0 with O, I such that $L_0=B(a_0, b_0)$ is a *mob* with the desired *kernel* $B(a, b)$ and with the multiplication defined as follows:

$xy=(a \smile x) \frown (b \smile y)$ for the fixed two elements a, b of L.

2. Lemma 1. *In a distributive lattice, $x \in B(a, b)$ if and only if $a \frown b \leqq x \leqq a \smile b$.*

Proof. This is proved in [1, Theorem 3].

Theorem 1. *In a distributive lattice L with O, I the B-topology*