47. On the Spectral-resolutions of Quasi-compact Elements in a B^*-algebra

By Yoshimichi MIBU

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Let $\mathfrak{B}$ be a complex Banach algebra with unit element e. We shall consider the inverse of $\lambda e-a$ as a function of λ for a fixed $a\in\mathfrak{B}$. According as $\lambda e-a$ is regular or singular in $\mathfrak{B}$, we say that λ belongs to the resolvent set $\rho(a)$ or the spectrum $\sigma(a)$ of a. And a non-void set σ is called a spectral set of a if σ is a subset of $\sigma(a)$ and σ is both open and closed in $\sigma(a)$. For λ in $\rho(a)$ the inverse of $\lambda e-a$ exists; it is denoted by $R(\lambda; a)$ and is called the resolvent of a. It is well known that the resolvent set $\rho(a)$ of a is open and in each of its components $R(\lambda; a)$ is a regular function of λ. The following theorem is proved in [1, pp. 105-107 (Theorems 5.11.1 and 5.11.2)].

Theorem A. Let $\sigma(a)=\bigcup_{i=1}^{n}\sigma_i$ where each σ_i is a spectral set of a and $\sigma_i\frown\sigma_j=0$ when $i\neq j$. Let us suppose that closed Jordan curves Γ_i, $i=1,2,\cdots,n$, satisfy the following conditions:

(i) For each $i(1\leqq i\leqq n)$ σ_i is contained in the open domain $\mathfrak{D}_i$ which is bounded by Γ_i.

(ii) $\mathfrak{D}_i\frown\mathfrak{D}_j=0$ when $i\neq j$.

(iii) Each Γ_i has the positive orientation, that is, the domain $\mathfrak{D}_i$ lies to the left of Γ_i.

If we define

$$(1)\qquad J_i=\frac{1}{2\pi i}\int_{\Gamma_i}R(\zeta; a)\,d\zeta \quad\text{and}\quad a_i=J_ia,\ i=1,2,\cdots,n,$$

then

$$(2)\qquad \sum_{i=1}^{n}J_i=e,\quad J_i^2=J_i,\quad J_iJ_j=\theta,\ i\neq j,\quad J_i\neq\theta, e,\quad \sum_{i=1}^{n}a_i=a.$$

Furthermore, the spectrum $\sigma(a_i)$ of a_i is σ_i in addition to $\lambda=0$, that is, $\sigma(a_i)=\sigma_i\smile\{0\}$. In particular, if σ_i is a single point λ_i, then an element $J_i(a-\lambda_ie)$ is quasi-nilpotent (or nilpotent).

First, under the assumption that $\mathfrak{B}$ is a commutative B^*-algebra (see Definition 1 below), we shall extend the above theorem to a case in which the spectrum $\sigma(a)$ of a consists of infinitely many components. Next, by using this extension, we shall offer a new proof for the spectral resolution theorem of compact normal operators in Hilbert spaces.

Definition 1. A Banach algebra $\mathfrak{B}$ in which every element a has an adjoint a^* with $(\alpha a+\beta b)^*=\bar{\alpha}a^*+\bar{\beta}b^*$, $(ab)^*=b^*a^*$, $a^{**}=a$, and