46. Some Remarks on Inner Product in Product Space of Unitary Spaces

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1. Let V be a unitary space over reals or complex numbers, and (x, y) be the inner product defined in it. It is known that inner product can be defined in the tensor product $V^r = V\otimes\cdots\otimes V$ (r factors in number) which satisfies: [2, 3]**)

$$(x_1\otimes x_2\otimes\cdots\otimes x_r,\ y_1\otimes y_2\otimes\cdots\otimes y_r)=(x_1, y_1)(x_2, y_2)\cdots(x_r, y_r).$$

This function, when restricted to the subspace of alternate elements $\mathcal{A}(V^r)$ and the subspace of symmetric elements $\mathcal{S}(V^r)$ of V^r, gives rise respectively to inner product of the space of exterior r-vectors $\Lambda^r(V)$ and $P^r(V)$ (to be defined below), since these spaces are respectively isomorphic to $\mathcal{A}(V^r)$ and $\mathcal{S}(V^r)$.

If u is the conjugate isomorphism between V and its dual (conjugate) space V^*, then

$$\langle x, u(y)\rangle=(x, y) \qquad \text{for all } x\in V,$$

where $\langle x, y^*\rangle$ is the pairing of V and V^* to scalars.

Denote by $u^r: V^r\to V_r = V^*\otimes\cdots\otimes V^*$ (r factors in number) the r-th tensor power of u, then u^r is an isomorphism between V^r and V_r and

$$u^r(x_1\otimes\cdots\otimes x_r)=u(x_1)\otimes\cdots\otimes u(x_r).$$

Moreover, if $\Lambda^r u: \Lambda^r(V)\to\Lambda^r(V^*)$ is the r-th exterior power of u, then $\Lambda^r u$ is an isomorphism between $\Lambda^r(V)$ and $\Lambda^r(V^*)$ and

$$(\Lambda^r u)(x_1\wedge\cdots\wedge x_r)=u(x_1)\wedge\cdots\wedge u(x_r).$$

As it is known that $(V^r)^*\approx V_r$ and $(\Lambda^r(V))^*\approx\Lambda^r(V^*)$, we can identify the isomorphic spaces.

Now, we propose to show:

Theorem 1. *u^r is the conjugate isomorphism between V^r and $V_r=(V^r)^*$, and $\Lambda^r u$ is the conjugate isomorphism between $\Lambda^r(V)$ and $\Lambda^r(V^*)=(\Lambda^r(V))^*$.*

Proof. For any $x_1\otimes\cdots\otimes x_r$ and $y_1\otimes\cdots\otimes y_r$ in V^r, we have

$$\begin{aligned}&\langle x_1\otimes\cdots\otimes x_r\quad u^r(y_1\otimes\cdots\otimes y_r)\rangle\\ &=\langle x_1\otimes\cdots\otimes x_r\quad u(y_1)\otimes\cdots\otimes u(y_r)\rangle\\ &=\langle x_1\ u(y_1)\rangle\cdots\langle x_r\ u(y_r)\rangle\end{aligned}$$

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**) In the sequel we follow the notation of S. S. Chern [2]. The number in bracket denotes the references at the end of this paper.