107. An Approximation Problem in Quasi-normed Spaces

By Kiyoshi Iséki

Kobe University

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Recently a linear metric space with a quasi-norm was considered by M. Pavel, S. Rolewicz and Konda. In this Note, we shall consider an approximation problem [1, p. 79] in quasi-normed spaces.

A quasi-normed space E of order s $(0 < s \leq 1)$ is a vector space over the real numbers on which is defined a non-negative real valued function $\|x\|$ called the *quasi-norm* such that

1) $\|x\| = 0$ implies $x = 0$,
2) $\|x+y\| \leq \|x\| + \|y\|$,
3) $\|\lambda x\| = |\lambda|^s \|x\|$,

where s is independent to x of E.

Let $x_1, x_2, \cdots, x_n$ be a set of linearly independent elements of E. We shall consider $\varphi(\lambda_1, \lambda_2, \cdots, \lambda_n) = \| y - \lambda_1 x_1 - \lambda_2 x_2 - \cdots - \lambda_n x_n \|$.
For $\lambda_1, \lambda_2, \cdots, \lambda_n; \mu_1, \mu_2, \cdots, \mu_n$, we have

$$| \varphi(\lambda_1, \lambda_2, \cdots, \lambda_n) - \varphi(\mu_1, \mu_2, \cdots, \mu_3) |$$

$$= \left| \left\| y - \sum_{i=1}^{n} \lambda_i x_n \right\| - \left\| y - \sum_{i=1}^{n} \mu_i y_i \right\| \right|$$

$$\leq \left\| \sum_{i=1}^{n} (\lambda_i - \mu_i) x \right\| \leq \sum_{i=1}^{n} \left\| (\lambda_i - \mu_i) x_i \right\|$$

$$= \sum_{i=1}^{n} |\lambda_i - \mu_i|^s \|x_i\|.$$

This shows that $\varphi(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is a continuous function of $\lambda_1, \lambda_2, \cdots, \lambda_n$. The function

$$\psi(\lambda_1, \lambda_2, \cdots, \lambda_n) = \| \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \|$$

on the unit sphere $\sum_{i=1}^{n} \lambda_i^2 = 1$ in n-dimensional space takes minimal value α, since the unit sphere is compact. The minimal value α is positive, since $x_1, x_2, \cdots, x_n$ are linearly independent.

Let M be a given positive number, and let

$$\left(\sum_{i=1}^{n} \lambda_i^2 \right)^{s/2} > \frac{1}{\alpha}(M + \|y\|),$$

then we have

$$\varphi(\lambda_1, \lambda_2, \cdots, \lambda_n) \geq \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| - \|y\|$$

$$= (\sum \lambda_i^2)^{s/2} \left\| \sum_{i=1}^{n} \frac{\lambda_i}{\sqrt{\sum_{k=1}^{n} \lambda_k^2}} x_i \right\| - \|y\|$$

$$\geq \left(\sum_{i=1}^{n} \lambda_i^2 \right)^{s/2} \alpha - \|y\| > M.$$