127. *Note on the Relations on Steenrod Algebra*

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The object of this note is to show some relations of binomial coefficients mod p where p is a prime, and using of them to show some relations on the Steenrod algebra. We shall use the results of José Adem.[1)]

1. Relations of binomial coefficients. Let $A_n=\sum_{i=0}^{n}\binom{n-i}{i}$, where n is any non-negative integer, so that

$$A_0=\binom{0}{0}=1,\ A_1=\binom{1}{0}+\binom{0}{1}=1,\ A_2=\binom{2}{0}+\binom{1}{1}+\binom{0}{2}=2,\cdots .$$

Generally

$$\begin{aligned}A_n&=\sum_{i=0}^{n}\left[\binom{n-i-1}{i}+\binom{n-i-1}{i-1}\right]\\&=\sum_{i=0}^{n-1}\binom{n-1-i}{i}+\binom{-1}{n}+\sum_{i=0}^{n-2}\binom{n-2-i}{i}+\binom{n-1}{-1}+\binom{-1}{n-1}\\&=A_{n-1}+A_{n-2}+(-1)^n+(-1)^{n-1}=A_{n-1}+A_{n-2},\end{aligned}$$

then we have inductively

$$A_{3t}\equiv 1,\ A_{3t+1}\equiv 1,\ A_{3t+2}\equiv 0 \bmod 2. \tag{1}$$

Let $B_b^a=\sum_{i=0}^{b}\binom{a+i(p-1)}{b-i}$ where a is any number and b is any non-negative integer, if $p=2$ it is easily recognized that $A_n=B_n^0$.

Then we will prove

$$B_b^a-B_{b-1}^a+\cdots+(-1)^iB_{b-i}^a+\cdots+(-1)^pB_{b-p}^a\equiv\binom{a}{b} \bmod p. \tag{2}$$

To prove this, deform B_b^a in two ways;

$$B_b^a=\binom{a}{b}+B_{b-1}^{a+(p-1)} \tag{3}$$

and

$$\begin{aligned}B_b^a&=\sum_{i=0}^{b}\left[\binom{a-1+i(p-1)}{b-i}+\binom{a-1+i(p-1)}{b-1-i}\right]=B_b^{a-1}+B_{b-1}^{a-1}\\&=\binom{p-1}{0}B_b^{a-(p-1)}+\cdots+\binom{p-1}{i}B_{b-i}^{a-(p-1)}+\cdots+\binom{p-1}{p-1}B_{b-(p-1)}^{a-(p-1)}\\&\equiv B_b^{a-(p-1)}+\cdots+(-1)^iB_{b-i}^{a-(p-1)}+\cdots+(-1)^{p-1}B_{b-(p-1)}^{a-(p-1)} \bmod p.\end{aligned} \tag{4}$$

Substituting the suitable expression (4) for the last term of (3) we have (2).

Hence from (4) and (2)

$$B_{b+p}^{a+(p-1)}\equiv(-1)^{p-1}B_b^a+\binom{a}{b+p} \bmod p. \tag{5}$$

Especially for any number a

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1) José Adem: The Relations on Steenrod Powers of Cohomology Classes, Algebraic Geometry and Topology, Princeton University (1957).