16. A Note on Algebras of Unbounded Representation Type

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Let A be a finite dimensional associative algebra over a field P. Recently J. P. Jans [2] has generalized R. M. Thrall's theorem concerned with algebras of unbounded representation type under a condition which can be applied to the case where P is an algebraically closed field. Jans's theorem is, however, a generalization of the condition given by T. Nakayama [4], which is also a generalization of the result of H. Brummund [1] (cf. also [3]). In this paper by a slight improvement of the proofs of Brummund the author will show that the theorem of Jans holds without the condition which is quoted above.

Except the following lemma which may be said to be our main device, the proofs in this paper are similar to those of Brummund, but for the sake of completeness and of reader's convenience we shall repeat them.

Lemma 1. *Let Q_κ and Q_λ be quasi-fields and M a left Q_κ-, right Q_λ-module*[1] *such that $(xu)y=x(uy)$ for $x\in Q_\kappa$, $y\in Q_\lambda$ and $u\in M$. If the left dimension of M over Q_κ as well as the right dimension of M over Q_λ is not less than two, then we can select two elements u and v such that they are independent (to each other) over Q_κ and Q_λ respectively.*

Proof. From the assumption we have

$$\begin{aligned} M&=Q_\kappa u+Q_\kappa u_1+\cdots \\ &=uQ_\lambda+u_2Q_\lambda+\cdots \quad . \end{aligned}$$

Then it is sufficient to prove the lemma for the case where $u_1\in uQ_\lambda$ and $u_2\in Q_\kappa u$. Put now $v=u_1+u_2$; then u and v satisfy our request. For if $v\in uQ_\lambda$, then $u_2\in uQ_\lambda$ and this is a contradiction. Similarly we can prove that $v\notin Q_\kappa u$ and the proof is completed.

Theorem 2. *If A has an infinite*[2] *two sided ideal lattice then A is of unbounded type.*

Proof. Let A have an infinite two sided ideal lattice; then the lattice is not distributive and contains a projective root (cf. [6], at the following diagram (projective root) B_1, B_2 and B are two sided ideals of A, and B_1 and B_2 cover B). Since the representations of the

1) All modules considered in this paper are unitary.

2) For the case of P being algebraically closed, in Theorem 2 "infinite two sided ideal lattice" can be replaced by "infinite one sided ideal lattice", but generally this replacement is not possible, for an algebra of strong left cyclic representation type may have an infinite left ideal lattice but is not of unbounded type (cf. [5, 7, 8]).