30. On Multi-valued Monotone Closed Mappings

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V. I. Ponomaleff [1] has defined the new space κX for T_1-space X. According to him the space κX is the set of all non-empty closed subsets of X and topology is defined as follows: for each point (F_0) of κX and for every neighborhood OF_0 of F_0 in X $D_1(OF_0)$ is the set of all closed subsets of X contained in OF_0 and these $D_1(OF_0)$ form the bases of the neighborhoods of (F_0) in κX. In our paper we shall use his definition for the topological space X (without T_1-axiom).

A multi-valued mapping f of a topological space X into a topological space Y is *monotone* if for each point x of X fx is closed in Y and for each pair of distinct points x and x' of X $fx \frown fx' = \phi$.

We use the definitions due to him: the continuity of a mapping f of X into Y is that for every point x of X and for each neighborhood Ofx of fx in Y there is a neighborhood Ox of x in X such that $fOx \subset Ofx$; the closedness of f is the closedness of the image of every closed subset of X; $\bar{f}$ is a one-valued mapping of X into κY which maps every point x of X to a point (fx) of κY.

Theorem 1. *If f is a one-valued closed continuous mapping of a topological space X onto a T_1-space Y, then the inverse mapping f^{-1} is a multi-valued monotone closed continuous mapping of Y onto X. Conversely, if g is a multi-valued monotone closed continuous mapping of a topological space X onto a topological space Y and if for every point y of Y $g^{-1}(y)=x$ such that $gx \ni y$, then g^{-1} is a one-valued closed continuous mapping of Y onto X.*

Proof. Since f is continuous, f^{-1} is closed, and since Y is T_1-space, f^{-1} is monotone. To prove that f^{-1} is continuous, let y be an arbitrary point of Y and $Of^{-1}(y)$ be an arbitrary neighborhood of $f^{-1}(y)$ in X. Since f is closed, there is an open inverse set $(Of^{-1}(y))_0$*) such that $f^{-1}(y) \subset (Of^{-1}(y))_0 \subset Of^{-1}(y)$. Then $V=f(Of^{-1}(y))_0$ is a neighborhood of y in Y such that $f^{-1}(V)=(Of^{-1}(y))_0 \subset Of^{-1}(y)$. This completes the proof that f^{-1} is a multi-valued monotone closed continuous mapping.

Conversely, let g be a multi-valued monotone closed continuous mapping of X onto Y. To show that g^{-1} is closed, let A be an arbitrary closed subset of Y. Since $g^{-1}(A)=\{x|gx \frown A \neq \phi;\ x \in X\}$, and if x_0 is an arbitrary point of $X-g^{-1}(A)$, then $gx_0 \frown A=\phi$; that is, $gx_0 \subset X-A$.

*) $(Of^{-1}(y))_0$ is the union of all $f^{-1}(p)$ $(p \in Y)$ such that $f^{-1}(p) \subset Of^{-1}(y)$.