29. On Pseudo-compact Spaces and Convergence of Sequences of Continuous Functions

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Among the results of this paper are some generalizations of results by Iséki [4] and Isiwata [5] concerning convergence properties for sequences of real-valued continuous functions. In particular it is shown that an arbitrary topological space X is pseudo-compact if and only if every sequence of continuous, real-valued functions which converges in some admissible (jointly continuous) topology for $C(X)$ converges uniformly. An additional characterization of pseudo-compact spaces is the following: If X satisfies the first axiom of countability, then X is pseudo-compact if and only if every sequence which converges in the compact-open topology on $C(X)$ converges uniformly.

We define several types of convergence for sequences of functions in $C(X)$, the ring of continuous, real-valued functions on an arbitrary topological space X.

(a) $\{f_n\}$ converges to f uniformly at each point of $X[f_n \to f(UP)]$ if, for each $x \in X$ and $\varepsilon > 0$ there is an integer N and a neighborhood U_x of x such that $|f_n(y)-f(y)|<\varepsilon$, whenever $y \in U_x$ and $n \geq N$.

(b) $\{f_n\}$ converges locally uniformly $[f_n \to f(LU)]$ if, for each $x \in X$, there is a neighborhood of x on which $\{f_n\}$ converges to f uniformly.

(c) $\{f_n\}$ converges strictly continuously to $f[f_n \to f(SC)]$ if, whenever $f(x_n)$ converges, $f_n(x_n)$ converges to the same limit. (See [4].)

(d) $\{f_n\}$ converges jointly to $f[f_n \to f(J)]$ if $\{f_n\}$ converges to f in some admissible topology for $C(X)$, i.e. a topology in which $(f, x) \to f(x)$ is a continuous mapping of $C(X) \times X$ into the real numbers. (See [5].)

In this paper we consider only continuous real-valued functions on a topological space X and by convergence we mean convergence to an element of $C(X)$.

THEOREM 1. *In any topological space X the following are equivalent.*

(i) *X is pseudo-compact.*

(ii) *Every sequence of continuous functions which converges uniformly at each point of X converges uniformly.*

(iii) *Every sequence of continuous functions which converges locally uniformly converges uniformly.*