28. On Orientable Manifolds of Dimension Three

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Let M be a closed orientable differentiable manifold of dimension 3 and f be a function on $M\times I$ where $I=[-1,1]$. Let x_i $(i=1,2,3)$ be a local coordinate system of M and t be the parameter varying on I. We write f_t instead of f when we consider that f is a function on M for fixed t. A point at which every first derivative of f_t with respect to x_i vanishes is called stational point and it is called ordinary stational point or super stational point according as: $\det\left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}\right)\neq 0$ or $\det\left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}\right)=0$.

If the origin $x_i=0$ $(i=1,2,3)$ is an ordinary stational point of f_0, in a neighborhood of this point f_t becomes

$$f_t=a(t)+\Sigma a_{ij}(t)x_i x_j$$

where $|t|$ is small and $\det(a_{ij}(0))\neq 0$.

And if $x_i=0$ is a super stational point of f_0, by a suitable coordinate system f_t is represented as

$$f_t=c+c_0t+\sqrt{-c_1t}\,x_1^2+c_2x_2^2+c_3x_3^2+\frac{1}{3}x_1^3$$

where $x_2=o(\sqrt{|t|})$ and $x_3=o(\sqrt{|t|})$. Here we can assume that all c_ν $(\nu=0,1,2,3)$ are not 0. Hence for a small $|t|$ we have two stational points $(0,0,0)$ and $(-2\sqrt{-c_1t},0,0)$ of f_t. At the point $(0,0,0)$ or $(-2\sqrt{-c_1t},0,0)$ f_t is represented as $c+c_0t+\sqrt{-c_1t}x_1^2+c_2x_2^2+c_3x_3^2$ or $c+c_0t-\sqrt{-c_1t}(x_1+2\sqrt{-c_1t})^2+c_2x_2^2+c_3x_3^2$ where all c_ν $(\nu=0,1,2,3)$ are not zero. We call a stational point to be type (μ) if the non-degenerate diagonal quadratic form in the Taylor's expansion of f_t at this point has μ negative terms. Suppose the above origin is type (μ) then $(-2\sqrt{-c_1t},0,0)$ is type $(\mu+1)$ and we call the super stational point $(0,0,0)$ of f_0 to be type $(\mu,\mu+1)$ or $(\mu+1,\mu)$ according as $c_1<0$ or $c_1>0$. We see easily that values of t on the locus of stational points take the minimums or the maximums at points of type $(\mu,\mu+1)$ or $(\mu+1,\mu)$.

Let D and D' be two solid spheres with n holes as Fig. 1 and σ a homeomorphism of ∂D to $\partial D'$ and $D\underset{\sigma}{\smile}D'$ the manifold defined by identifying ∂D and $\partial D'$ by σ.

Now we consider the necessary and sufficient condition so that $D\underset{\sigma}{\smile}D'$ is diffeomorphic with $D\underset{\tau}{\smile}D'$. Clearly we can construct a function g on $D\underset{\sigma}{\smile}D'$ satisfying the following conditions.

a) $g<0$ in $D-\partial D$, $g=0$ on ∂D and $g>0$ in $D'-\partial D'$.