50. On Characterizations of Projection Operators

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Let R be a lattice ordered linear space. A linear manifold M of R is said to be *normal*, if for any $a\in R$ we can find $x, y\in R$ such that

$$a=x+y \quad x\in M, \quad y\in M^{\perp}=\{y;\ x\perp y \text{ for } x\in M\}.$$

Such x depends only on a. So putting $Ta=x$ we can define an operator T from R to M. This operator is called a *projection operator* (cf. H. Nakano: Modulared Semi-ordered Linear Space, Tokyo (1950)).

Here, we will consider some characterizations of projection operators.

Theorem 1. *A linear operator T on R is a projection operator, if and only if it satisfies* (1), (2).

(1) $$T(Tx)=Tx$$

(2) $$0\leqq Tx\leqq x \qquad \textit{for all } x\geqq 0.$$

Proof. Every projection operator is always linear and satisfies (1), (2) (cf. H. Nakano: Modulared Semi-ordered Linear Space, Tokyo (1950)).

Now, we suppose that a linear operator T satisfies conditions (1), (2). Putting $T^{\perp}=I-T$, $T^{\perp}$ is obviously linear and satisfies conditions (1), (2) too. When we consider two subsets of R

$$A=\{x;\ Tx=0\} \quad \text{and} \quad B=\{x;\ T^{\perp}x=0\},$$

we have $A=T^{\perp}R$, $B=TR$, because

$$T(T^{\perp}a)=T(a-Ta)=Ta-T(Ta)=Ta-Ta=0,$$

for any $a\in R$, and hence $T^{\perp}a\in A$. On the other hand, we see

$$a=a-Ta=T^{\perp}a,$$

for every $a\in A$, therefore $A=T^{\perp}R$. We obtain $B=TR$ likewise.

Every linear operator T, subject to the condition (2), satisfies

$$T(x\overset{\smile}{\frown}y)=Tx\overset{\smile}{\frown}Ty.$$

Because we see first obviously

$$Tx_{\frown}Ty\geqq T(x_{\frown}y).$$

On the other hand, we have

$$x=Tx+T^{\perp}x\geqq Tx_{\frown}Ty+T^{\perp}(x_{\frown}y),$$
$$y=Ty+T^{\perp}y\geqq Tx_{\frown}Ty+T^{\perp}(x_{\frown}y)$$

and hence

$$x_{\frown}y\geqq Tx_{\frown}Ty+T^{\perp}(x_{\frown}y),$$

that is,

$$T(x_{\frown}y)\geqq Tx_{\frown}Ty.$$

Therefore

$$T(x_{\frown}y)=Tx_{\frown}Ty.$$

Then we find easily

$$T(x^{\smile}y)=Tx^{\smile}Ty,$$