17. On Tonelli's Theorem concerning Curve Length

By Kanesiroo ISEKI
Department of Mathematics, Ochanomizu University, Tokyo
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1. **Introduction.** Let us consider a plane parametric curve (not necessarily continuous) given by the equation $\varphi(t)=\langle x(t), y(t)\rangle$, where the variable t ranges over the real line $\boldsymbol{R}$. We assume that this curve is locally rectifiable, i.e. that its arc length $s(I)$ is finite for any closed interval I in $\boldsymbol{R}$. We are interested in the problem of expressing the length by means of the derivatives $x'(t)$ and $y'(t)$. Of course this is easily solved when, in particular, the curve is continuously differentiable, since we have then, for every I, the well-known formula

$$s(I)=\int_I \sqrt{x'(t)^2+y'(t)^2}\, dt. \tag{1}$$

In the general case, however, the same problem shows itself far more complicated and was not solved until Tonelli proved the following decisive result: *we have the relation* $s'(t)^2=x'(t)^2+y'(t)^2$ *for almost every point* t *of* $\boldsymbol{R}$, *and the integral on the right of* (1) *does not exceed* $s(I)$ *for any closed interval* I, *the equality* (1) *holding if and only if both the functions* $x(t)$ *and* $y(t)$ *are absolutely continuous on* I (see Saks [3], p. 123).

Now Tonelli's theorem, though without doubt fautless in its own way, cannot nevertheless be regarded, so far as it goes, as a complete and final solution of the problem under consideration, in the following one point: it gives us no insight, even when the curve is continuous, into the nature of the difference between the arc length $s(I)$ and the square-root integral. It is the main object of the present note to remedy this defect by obtaining, at least for continuous curves, a supplement to Tonelli's theorem which resembles in enunciation the decomposition formula of de la Vallée Poussin (*vide* Saks, p. 127).

2. **Heuristic considerations.** Retaining the notation of the introduction, let us write E_x for the Borel set of the points t for which $x'(t)=\pm\infty$, and let E_y be defined correspondingly. According to de la Vallée Poussin's theorem (*loc. cit.*) we have, for every bounded Borel set A at whose points t the curve $\varphi(t)$ is continuous,

$$x^*(A)=x^*(AE_x)+\int_A x'(t)dt$$

and a similar relation for y^* (the set E_x being replaced by E_y, needless to say), where x^* and y^* represent the outer measures of Carathéodory induced by $x(t)$ and $y(t)$ respectively. This at once suggests us the